A logic toolbox for modeling knowledge and information in multi-agent systems and social epistemology

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by

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Abstract

This dissertation consists of a collection of papers on pure and applied modal logics preceded by an introduction that ties them together. The papers and the variety of topics discussed in them can be viewed as a contribution to a logic toolbox. More specifically, a logic toolbox streamlined to model information, knowledge, and beliefs. The logic tools have many applications in various fields such as computer science, philosophy, mathematics, linguistics, economics, and other social sciences, but in this thesis the focus will be on their application in social epistemology and multi-agent systems.

After the introduction, which outlines the logic toolbox and some of its applications, follow four technical chapters that expand the logic toolbox (chapters 2, 3, 4, and 5) and two chapters that show the logic tools at work (chapters 6 and 7). In Chapter 2 a many-valued hybrid logic is introduced and a sound, complete, and terminating tableau system is given. Chapter 3 is a supplement to Chapter 2 and discusses various alternative definitions of the semantics of the hybrid part of the many-valued hybrid logic. Chapter 4 combines public announcement logic with hybrid logic and gives a sound and complete axiomatisation of the logic as well as discussing extensions with other modalities such as distributed knowledge. In Chapter 5, terminating tableau systems using reduction axioms as rules are given for standard dynamic epistemic logic as well as the hybrid public announcement logic of Chapter 4. Chapter 6 uses a dynamic epistemic logic to model the phenomenon of pluralistic ignorance. Finally, Chapter 7 uses first-order logic and description logic to discuss foundational issues in knowledge representation of regulatory relations in biomedical pathways.

This dissertation makes numerous contributions to the field of which the following three are the most important:

- A detailed investigation of the combination of hybrid logic and public announcement logic, showing in particular that the proof theoretic advantages of hybrid logic (such as automatic completeness with pure

formulas) extend to hybrid public announcement logic (chapters 4 and 5).

- An approach to constructing terminating tableau systems for hybrid logics is shown to be extremely general in the sense that it extends to both many-valued hybrid logic (Chapter 2), dynamic epistemic logic, and hybrid public announcement logic (Chapter 5).
- This dissertation touches upon the issue of how knowledge and beliefs of a group of agents relate to the knowledge and beliefs of the individuals of the group. How knowledge or beliefs of the individual agents can be aggregated to yield group knowledge or group beliefs is studied in Section 1.3.3 of the introduction, and how information and beliefs flow among a group of agents involved in the phenomenon of pluralistic ignorance is studied in Chapter 6.

Resumé

Denne afhandling består af en samling artikler, der omhandler teoretisk og anvendt modallogik, bundet sammen af en forudgående indledning. Artiklerne og de deri behandlede emner kan ses som et bidrag til en logik-værktøjskasse. Mere præcist: en logik-værktøjskasse trimmet til at modellere information, viden og formodninger. Logik-værktøjet har mange anvendelser i discipliner såsom datalogi, filosofi, matematik, lingvistik, økonomi og andre samfundsvidenskaber, men i denne afhandling er fokusset påanvendelser i social epistemologi og multi-agent systemer.

Efter introduktionen, der skitserer logik-værktøjskassen og nogle af dens anvendelser, følger fire tekniske kapitler, som udvider denne værktøjskasse (kapitlerne 2, 3, 4 og 5) og to kapitler, der viser logik-værktøjerne i arbejde (kapitlerne 6 og 7). I kapitel 2 introduceres en mange-værdi hybridlogik og et sundt, fuldstændigt og terminerende tableau-system given for den. Kapitel 3 er et tillæg til kapitel 2 og diskuterer forskellige alternative definitioner af semantikken for den hybride del af mange-værdi hybridlogikken. Kapitel 4 kombinerer "offentlig annoncerings"-logik med hybridlogik, indfører en sund og fuldstændig aksiomatisering af logikken og diskuterer desuden udvidelser med andre modaliteter såsom distribueret viden. I kapitel 5 præsenteres terminerende tableau-systemer, hvor reduktionsaksiomer bruges som regler for standard dynamisk epistemisk logik og den offentlig annoncerings-logik fra kapitel 4. Kapitel 6 bruger en dynamisk epistemisk logik til at modellere fænomenet "pluralistisk ignorance". Endelig bruger kapitel 7 førsteordenslogik og beskrivelseslogik til at diskutere fundamentet for vidensrepræsentation af regulatoriske relationer i biomedicinske netværk.

Afhandlingen kommer med adskillige videnskabelige bidrag. De tre vigtigste er dog som følger:

- En detaljeret undersøgelse af kombinationen af hybridlogik og offentlig annoncerings-logik, som særligt påviser, at de bevisteoretiske fordele ved hybridlogik (såsom automatisk fuldstændighed med rene former) kan

overføres til hybrid offentlig annoncerings-logik (kapitlerne 4 og 5).

- En metode til at konstruere terminerende tableau-systemer for hybridlogik vises at være ekstrem generel i den forstand, at den kan udvides til bade mange-værdi hybridlogik (kapitel 2) og dynamisk epistemisk logik og offentlig annoncerings-logik (kapitel 5).
- Endelig berører denne afhandling emnet, om hvordan en gruppes viden og formodninger relaterer sig til gruppens individers viden og formodninger. Hvordan individuelle agenters viden eller formodninger kan aggregeres til gruppeviden eller gruppeformodninger diskuteres i afsnit 1.3.3 af introduktionen. Hvordan information og formodninger flyder mellem en gruppe af agenter involveret i fænomenet pluralistisk ignorance udforskes i kapitel 6.

Acknowledgments

In the process of writing this dissertation and conducting the research it reports, I have received support and advice from a great number of people. At the end of each chapter, the people who have specifically influenced the particular research in that chapter are acknowledged. In addition, however, I would like to take this opportunity to give a general thanks to a number of people.

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During my PhD studies I also had the chance to spend one year at the Institute of Logic, Language and Computation at the University of Amsterdam and in this connection I have many people to thank. First of all, Frank Veltman for being my official host at the ILLC. Secondly, the teachers, organizers, and participants of the many courses, seminars, workshops, conference, and study groups I attended in Amsterdam. Finally, a thanks goes out to the many people who made my stay in Amsterdam an exceptionally enjoyable experience

- again, mentioning you all would be exceedingly difficult.

Besides of Amsterdam, I have attended many conferences, workshops, and summer schools, where I met numerous interesting people and had countless inspiring discussions. Especially the European Summer School in Logic, Language and Information (ESSLLI) has introduced me to many interesting people and taught me a great deal. Thanks to Vincent Hendricks for bringing ESSLLI to Copenhagen last year, and thanks to the other organizers and student session co-chairs for making it a great experience. Also, thanks to the organizers of the PhDs in Logic workshops, which I much enjoyed as well.

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Jens Ulrik Hansen
Copenhagen, September 2011

Contents

Chapter 1

Introduction

This thesis consists of a collection of papers on pure and applied logic that covers a variety of technical results as well as a few applications. The primary focus of the thesis is on technical issues invovled in expanding the logics and their proof theory. There are a number of topics, methods, and ideas that are shared by the papers and the aim of this introduction is to show how they can all be viewed as part of a single project, namely the project of expanding the logic toolbox for modeling knowledge and information in multi-agent systems and social epistemology. This endeavor not only presupposes a certain view of what logic is and can be used for, but also a certain view on what knowledge and information are, and how agents, be it humans, computer programs, or robots, represent, process, and reason about knowledge and information. Therefore, this introduction is intended to clarify the view of knowledge and information adopted in this endeavor, to clarify how this view results in applications within computer science and philosophy, and finally to clarify the logical frameworks on which this thesis is based.

This introduction starts the unlocking of the toolbox by introducing the logics appearing in this thesis, which include modal logic, hybrid logic, description logic, epistemic logic, dynamic epistemic logic, and many-valued logic. The unlocking continues with a discussion of the proof theory of several of these logics. Following the listing of the tools, the fields in which they can be applied are outlined. This outline includes an introduction to information, knowledge, beliefs and the problems they raise within areas of philosophy and computer science as well as a discussion of how logic can be of assistance.[1]

[1] A word of warning for the computer scientist regarding the toolbox metaphor: The aim of this thesis has been to develop a conceptual toolbox, not actual tools in form of computer systems or programs.

1.1 The logic toolbox I: Modal logic and some of its friends

Logic is an old subject within philosophy and has traditionally been defined as the systematic study of valid reasoning. The central object of study was arguments and their logical forms, based on which a notion of what constitutes a valid argument could be properly defined: an argument being valid if the truth of the premises ensures the truth of the conclusion. However, since the great influence of computer science on logic, both as providing applications and new theoretical concepts, the subject has become much broader. In general, there is no doubt that logic has become a broader field due to its many applications within fields such as computer science, artificial intelligence, linguistics, economics, and mathematics.

There is much that can be said about the history of logic, and the debate of what constitutes the subject today is no trivial discussion either. Instead of going further into these matters, an explanation of the view of logic adopted in this thesis will be laid out. Here logic will be viewed simply as a modeling tool and not as merely a study of arguments' forms and notions of validity. Logic can be viewed as just another formal/mathematical framework which can be used to model various phenomena, such as computations, natural language, or rational interactions.[2] The view that logic is a formal tool, useful for modeling various scenarios, is not claimed to be the only ideal view of logic, but it fits very well with the current state of formal and social epistemology as well as multi-agent systems and artificial intelligence. Within these fields logic can provide a useful toolbox that certainly justifies further study.

However, this thesis just touches upon a corner of this enormous toolbox. Nevertheless, all chapters of this thesis involve some kind of modal logic (Chapter 7 only briefly). Therefore, modal logic, and some of its extensions, will be discussed specifically in the rest of this section. In the discussion, focus will be on the semantic aspects of the logics, but in Section 1.2 the syntactic or proof theoretic aspects of some of the logics will be elaborated.

[2]Viewing logic as a modeling tool immediately raises two questions: What are we trying to model and how adequate is the modeling? The main focus of this thesis is on the technical issues involved in expanding existing tools and thus the adequateness of the logics developed will be given very little attention. Examples of what logic in general can be used to model is given Section 1.3. The chapters 6 and 7 are also examples of what logic can be used to model, and the adequateness of the particular logics will be discussed in these chapters, especially in Chapter 7.

1.1.1 Modal logic

The beginning of modal logic is attributed to Aristotle [34] and ever since then, modal logic has been a part of philosophy. Traditionally conceived, modal logic is the study of reasoning with modal expressions such as “necessarily”, “possibly”, “must”, “can” etc., which all moderate the truth values of statements. Statements might not only be true they may be true with different modes, for instance necessarily true, possibly true etc.. More broadly conceived modal logic also deals with other modalities than “it is necessary that” and “it is possible that” (alethic modalities), for instance “it will be the case that”, “it has always been the case that” (temporal modalities), “it is obligatory that”, “it is permitted that” (deontic modalities) “a knows that”, “a doubts that”, “a believes that” (epistemic and doxastic modalities). Modal logic as the study of reasoning with such modal expressions is more or less the standard view on what modal logic is within philosophy [64, 72, 131, 68].

However, modal logic has moved past the borders of philosophy and is now broadly used within computer science, artificial intelligence, linguistics, mathematics, economic game theory and other fields. For instance, new modalities have come in from computer science such as “after all runs of the program a”, “once process a is started, eventually”. Furthermore, a mathematical generalization of modal logic has also occurred: one of the standard references on modal logic, [27], describes modal logic as a logic for reasoning about general relational structures from an internal, local perspective. Moreover, [27] notes that other logics can be used to reason about relational structures and modal logic can reason about other things than relational structures, which leads to two ways of extending the field of modal logic even further: New logics to reason about relational structures can be developed and existing modal logics can be used to reason about other kind of mathematical structures. Where the view from philosophy of modal logic starts from modal expressions appearing in natural or formal languages, the view of modal logic as reasoning about relational structures focuses on how modalities and their semantics are defined mathematically, and therefore becomes a purely mathematical study.

All this goes to show that modal logic is a wide subject that appears in several fields and lends itself to many different approaches. The broadness of the subject is also witnessed by the recent handbook of modal logic [31]. In conclusion, modal logic is one of the fastest growing corners of the logic toolbox, both because of technical advances in the mathematical/computational theory of modal logic and because of the many new applications within numerous other fields. In the remainder of this section, standard modal logic will be described in more detail. After introducing modal logic broadly, hybrid logic will be introduced in Section 1.1.2, description logic in Section 1.1.3, epistemic logic in Section 1.1.4, dynamic epistemic logic in Section 1.1.5, and many-valued logic in Section 1.1.6.

Viewing modal logic as a logic to reason about relational structures is an idea that has had significant influence on modern modal logic. Using relational structures to interpret the statements of modal logic is an idea going back to the middle of the last century and is usually ascribed to Saul Kripke, even though others had similar ideas at the time. However, Kripke's formulation remains the most general and clear, which is why the relational semantics is often referred to as Kripke semantics. In the following it will either be referred to as Kripke semantics or possible worlds semantics. For more on the early development of the semantics of Kripke and others see [74].

The idea behind Kripke semantics is based on Leibniz's notion of possible worlds. The world we live in (the actual world) could have been different in many ways corresponding to different possible worlds. Then, something is necessarily true if it is true in all possible worlds and something is possible if there is a possible world in which it is true. The real generality comes from not taking a necessary statement to be true precisly if it is true in *all* possible worlds, but if it is true in all possible worlds that are possible relative to the current world. This means that statements are no longer just true or false, they are always true or false relative to a possible world. These intuitions will now be made mathematically precise.

In order to make the possible world semantics of modal logic mathematically precise, a formal language has to be specified first. The formal language is an extension of the standard propositional language containing the logical constants $\wedge$, $\vee$, $\neg$, $\rightarrow$, and $\leftrightarrow$. As basics an countable infinite set of propositional variables PROP will be assumed, and the elements of PROP will normally be denoted by $p, q, r, \ldots$. These variables can range over any basic propositions. The formulas of standard modal logic are then inductively defined by:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \rightarrow \varphi) \mid (\varphi \leftrightarrow \varphi) \mid \Diamond\varphi \mid \Box\varphi,$$

where $p \in \mathsf{PROP}$. The meaning of this is that all $p \in \mathsf{PROP}$ are formulas, if φ is a formula then $\neg\varphi$ is a formula, if φ and ψ are formulas then $(\varphi \wedge \psi)$ is also a formula, and so on. The outer parentheses of a formula will normally be omitted. In most of the logics that will be considered the connectives $\vee$, $\rightarrow$,

$\leftrightarrow$ will be definable from $\wedge$ and $\neg$, and therefore only these two connectives will be used in specifying formal languages. Furthermore, the $\Diamond$ and $\Box$ will also be definable from each other by $\Diamond = \neg\Box\neg$ and $\Box = \neg\Diamond\neg$, and as a result normally only one of them will be used when specifying formal languages from now on.[3] Finally, in some cases $\top$ and $\bot$ will be used as atomic propositions referring to a tautology (for instance $p \vee \neg p$) and a contradiction (for instance $p \wedge \neg p$).

The alethic reading of the modal formula $\Box\varphi$ is "it is necessarily true that φ" and the reading of $\Diamond\varphi$ is "it is possible that φ". Taking necessarily true to mean "true in all possible worlds", a formal semantics that reflects this can be provided for the language. A *possible world model* (or just a *model*) $\mathcal{M}$ is a tuple $\langle W, R, V\rangle$, where W is a non-empty set, R is a binary relation on W, and V is a function $V : \mathsf{PROP} \to \mathcal{P}(W)$. W is referred to as the set of possible worlds (or states), and R is the accessibility relation; $R(w, v)$ will be read as "the world v is accessible from the world w".[4] The pair $\langle W, R\rangle$ is also called a frame, and if $\mathcal{M}$ is $\langle W, R, V\rangle$, $\mathcal{M}$ is said to be based on the frame $\langle W, R\rangle$. Finally, V is a valuation that specifies the truth-value of every propositional variable at every possible world in W, hence $V(p)$ will be conceived as the set of possible worlds where p is true.

As already mentioned, modal formulas are always true (or false) relative to a world, which is reflected in the basic semantic relation $M, w \models \varphi$ (reading φ is true at the world w in the model $\mathcal{M}$). This relation is defined inductively for any model $\mathcal{M} = \langle W, R, V\rangle$, any world $w \in W$, and any modal formula φ, by:

$$
\begin{array}{lll}
\mathcal{M}, w \models p & \text{iff} & w \in V(p) \\
\mathcal{M}, w \models \neg\varphi & \text{iff} & \text{it is not the case that } \mathcal{M}, w \models \varphi \\
\mathcal{M}, w \models \varphi \wedge \psi & \text{iff} & \mathcal{M}, w \models \varphi \text{ and } \mathcal{M}, w \models \psi \\
\mathcal{M}, w \models \varphi \vee \psi & \text{iff} & \mathcal{M}, w \models \varphi \text{ or } \mathcal{M}, w \models \psi \\
\mathcal{M}, w \models \varphi \to \psi & \text{iff} & \mathcal{M}, w \models \varphi \text{ implies that } \mathcal{M}, w \models \psi \\
\mathcal{M}, w \models \varphi \leftrightarrow \psi & \text{iff} & \mathcal{M}, w \models \varphi \text{ if, and only if } \mathcal{M}, w \models \psi \\
\mathcal{M}, w \models \Diamond\varphi & \text{iff} & \text{there exists a } v \in W, \text{ such that } R(w, v) \text{ and } \mathcal{M}, v \models \varphi \\
\mathcal{M}, w \models \Box\varphi & \text{iff} & \text{for all } v \in W, \text{ if } R(w, v) \text{ then } \mathcal{M}, v \models \varphi.
\end{array}
$$

[3]In the many-valued logics of chapters 2 and 3, $\wedge$, $\vee$, $\to$, $\Box$, and $\Diamond$ will all be included since none of them will be definable from the others, which is natural in a many-valued setting. Many-valued logics will also be discussed further in Section 1.1.6.

[4]Instead of writing $R(w, v)$, the notation wRv or $(w, v) \in R$ will also be used on occasions.

A formula φ is said to be satisfiable if there is a model $\mathcal{M} = \langle W, R, V \rangle$ and a $w \in W$ such that $\mathcal{M}, w \models \varphi$. φ is said to be true in a model $\mathcal{M} = \langle W, R, V \rangle$ (written $\mathcal{M} \models \varphi$), if $\mathcal{M}, w \models \varphi$ for all $w \in W$. Formula φ is said to be valid on a frame $\mathcal{F} = \langle W, R \rangle$ (written $\mathcal{F} \models \varphi$), if $\mathcal{M} \models \varphi$ for all models $\mathcal{M}$ based on $\mathcal{F}$. If F is a class of frames then φ is said to be valid on F if $\mathcal{F} \models \varphi$ for all $\mathcal{F} \in \mathsf{F}$. Finally, a formal φ is said to be valid if φ is valid on the class of all frames.

The possible world semantics for modal logic can also be used for other modalities than just the alethic ones. In the various applications of modal logic, the possible worlds can be many things such as points of time, worlds conceived epistemically possible by agents, and states in a computation. The accessibility relations then represent the flow of time, the epistemic indistinguishability between worlds, or the transition of a computation. The epistemic interpretation of modal logic and its possible world semantics is adopted in chapters 4, 5, and 6 and will also be further discussed in Section 1.1.4. The possible world semantics for modal logic is by far the most common one, however, other semantics are possible, see [30]. In this thesis only modal logics with possible world semantics will be investigated.

Note how the definition of the semantics of $\Box$ and $\Diamond$ uses the quantifiers "there exists" and "for all", which allows modal logic to be viewed as a fragment of first-order logic.[5] However, viewing modal logic this way leads to the natural questions of whether there are other fragments between standard modal logic and first-order logic. The answer is affirmative and hybrid logic is precisely the study of a family of such fragments.[6]

1.1.2 Hybrid logic

Modal logic talks about relational structures in an internal local way, without explicitly mentioning the worlds of the models. This results in clear and simple languages, but it also limits their expressive power. In the temporal reading of modal logic one can express statements like "in the future it will rain" and "it is always going to be the case that grass is green", but one cannot express statements like "it is the 1st of March 2011" or "the meeting is at 11 o'clock on the 24th of April 2011". However, this kind of reference to specific points in time seems very natural in temporal reasoning, and should therefore be

[5]See [27] for more on the relations between modal and first-order logic.

[6]There are other possible extensions of modal logic that goes beyond mere first-order logic and well into second-order logic. Such extensions will not be discussed in this thesis, with the small exception of the common knowledge modality shortly mentioned in Chapter 4.

possible in modal logic, at least when given a temporal interpretation. An extension of modal logic that allows such references to specific possible worlds is exactly what hybrid logic is.

Hybrid logic is a term used to refer to a broad family of logics living between standard modal logic and first-order logic. Still, they almost all include a special kind of propositional variables called *nominals*. By demanding that each nominal is true in exactly one world, they provide a way of referring to specific worlds in models (or specific points in time in the temporal reading). This way of using special propositional variables to refer to/denote worlds goes back to Arthur Prior in the 1950s and his work on temporal logic. However, hybrid logic was later independently invented in the 1980s by "the Sofia school" in Bulgaria (George Gargov, Solomon Passy and Tinko Tinchev). Since then, much has happened in hybrid logic and for more on the history of hybrid logic see [8, 26, 41, 42].

Besides nominals, hybrid logic normally also includes satisfaction operators. Given a nominal i, a satisfaction operator $@_i$ is included, which allows for the construction of formulas of the form $@_i\varphi$. The reading of $@_i\varphi$ is "φ is true at the world denoted by i". The name "satisfaction operator" comes from the fact that this operator actually internalizes the semantic satisfaction relation "$\models$".

In addition to nominals and satisfaction operators, the hybrid logic family contains several other possible extensions. One extension is to include the global modality that allows for quantification over the entire set of possible worlds of a model, in its existential form denoted by "E" and in its universal form denoted by "A". Another extension is to add the "downarrow binder" $\downarrow$ that allows for the construction of formulas of the form $\downarrow i.\varphi$. The intuition behind the formula $\downarrow i.\varphi$ is that: "$\downarrow i.\varphi$ is true at a world w, if φ is true at w when i denotes the world w". Thus, the job of $\downarrow i.$ is to name the current world i. Other extra machinery can be added as well, but nominals, satisfaction operators, the global modality, and the downarrow binder are the only hybrid machinery used in this thesis. For further extensions, or a detailed introduction to hybrid logic in general, see [8, 41, 42].

It is time to make the extensions mentioned formal. In addition to the set of propositional variables PROP an countable infinite set of nominals NOM is assumed, such that $\mathsf{PROP} \cap \mathsf{NOM} = \emptyset$. The elements of NOM will normally be denoted by $i, j, k, \ldots$. The formulas of full hybrid logic are then inductively defined by (taking $\vee$, $\rightarrow$, $\leftrightarrow$, $\Diamond$ to be defined as described in the last section):

$$\varphi ::= p \mid i \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi \mid @_i\varphi \mid E\varphi \mid {\downarrow}i.\varphi,$$

where $p \in \mathsf{PROP}$, and $i \in \mathsf{NOM}$.[7] The global modality in its universal form A will be defined by $\neg E \neg$. Various weaker hybrid logics can be obtained by leaving out some of the machinery (like the downarrow binder, which will only be included in Chapter 4).

Possible world models for hybrid logic are the same as for standard modal logic, i.e. of the form $\mathcal{M} = \langle W, R, V\rangle$, but now $V : \mathsf{PROP} \cup \mathsf{NOM} \rightarrow \mathcal{P}(W)$ is required to satisfy that for all $i \in \mathsf{NOM}$, $V(i)$ is a singleton set. With this small change, the semantics of the new elements of the language can be defined. This is defined inductively for any model $\mathcal{M} = \langle W, R, V\rangle$, any world $w \in W$, and any hybrid formula φ, by:

$$\begin{array}{lll}
\mathcal{M}, w \models i & \text{iff} & \{w\} = V(i) \\
\mathcal{M}, w \models @_i\varphi & \text{iff} & \mathcal{M}, v \models \varphi, \text{ where } v \in V(i) \\
\mathcal{M}, w \models E\varphi & \text{iff} & \text{there exists a } v \in W, \text{ such that } \mathcal{M}, v \models \varphi \\
\mathcal{M}, w \models {\downarrow}i.\varphi & \text{iff} & \langle W, R, V'\rangle, w \models \varphi, \text{ where } V' \text{ is like } V, \\
 & & \text{except that } V'(i) = \{w\}.
\end{array}$$

The notions of validity and truth in a model are the same as for standard modal logic. The formula $@_i\varphi$ expresses that φ is true at the world named by i and this is exactly what is meant by the claim that $@_i$ internalizes the satisfaction relation $\mathcal{M}, w \models \varphi$. Furthermore that the world named by j is accessible from the world named by i can also be internalized into the language as $@_i\Diamond j$. These will be key properties when discussing the proof theory of hybrid logic in Section 1.2.2.

From the semantics it can be shown that the formula $@_i\varphi$ is definable as $E(i \wedge \varphi)$ or $A(i \rightarrow \varphi)$ and thus satisfaction operators are superfluous in the presence of the global modality. However, one of the discoveries of this thesis is that the formulas $E(i \wedge \varphi)$ and $A(i \rightarrow \varphi)$ are not necessarily equivalent in all versions of hybrid logic, such as the many-valued hybrid logics of Chapter 3 or the hybrid public announcement logic of Chapter 4. Furthmore, in Chapter 3 a logic is presented for which $@_i\varphi$ is neither definable as $E(i \wedge \varphi)$ nor as $A(i \rightarrow \varphi)$.

[7] When introducing the downarrow binder in Chapter 4 an extra set of state variables is included for the downarrow binder to quantify over. The reason is that it allows for a distinction between nominals and names introduced by the downarrow binder, which is useful in developing the proof theory in Chapter 4.

The extra added machinery of hybrid logic makes properties of models and frames expressible that were not expressible in standard modal logic. It is worth noticing that hybrid logic increases the expressive powers of modal logic, sometimes even without an extra cost of increased complexity, see [7]. There is much more to say about the expressive power of hybrid logic, but nothing more will be said here. The chapters of this thesis dealing with hybrid logic contain brief discussions of the expressivity of the presented hybrid logics. For more on the expressivity of hybrid logic in general see [8, 146].

Another great advantage of hybrid logic is its nice and simple proof theory. This has driven much recent research in hybrid logic, see [42]. The proof theory of hybrid logic plays an important role in this thesis as chapters 2, 4, and 5 provide new proof theory for various hybrid logics. The proof theory of hybrid logic will be properly introduced in Section 1.2.2.

Hybrid logic will appear in chapters 2, 3, 4, and 5 in slightly different versions than the one presented here. First of all, due to the combination with the public announcement operator[8] in chapters 4 and 5 nominals will not be required to be true in *exactly one world*, but in *at most one world*. The reason for this modification to standard hybrid logic semantics is further described in Chapter 4. Chapters 2 and 3 also contain a modification to the standard hybrid logic semantics because the logics of these chapters are many-valued logics. In a many-valued setting a statement like "the nominal i is true in exactly one world" becomes ambiguous and the entire purpose of Chapter 3 is to investigate different possible semantics for nominals in a many-valued setting.

The study of hybrid logic and modal logic has identified several interesting fragments of first-order logic that turn up in other connections. Description logic, which is a family of logics designed for knowledge representations, corresponds to many of the same fragments of first-order logic as hybrid logic and modal logic. Moreover, description logic is interesting in its own right in relation to modeling knowledge and information, and will briefly be used in Chapter 7. Therefore a short elaboration of description logic will now be given.

[8]The public announcement operator comes from dynamic epistemic logic and will be discussed in more details in Section 1.1.5.

1.1.3 Description logic

Description logic is the result of a long development in formal knowledge representation that has turned out to be a reinvention of not just modal logic, but hybrid logic, [12]. However, there is much more to description logic than just being a notational variant of some hybrid or modal logics. The entire intuitions behind the formulas are different and the way the logic is used for knowledge representation gives rise to a new large family of reasoning tasks other than the standard search for validities. This section will elaborate on the intuitions behind description logic and present a little of the formal syntax and semantics of the logic. How description logic is used in knowledge representation will be the topic of Section 1.3.2.1.

Two kinds of knowledge about the world (or a specific domain) can be distinguished, namely *terminological knowledge* and *world assertions*. Terminological knowledge is knowledge about the structure of the concepts used to describe the world, whereas world assertions are descriptions of which individuals exist and which concepts they satisfy. Thus, the basics are no longer propositions, but concepts or concept descriptions.

Formally, concept descriptions are built up from a set of atomic concepts (usually denoted by A or B) and a set of atomic roles (usually denoted by R) using concepts or role constructors. Given these two sets, concept descriptions are built up by the following syntax:

$$C ::= A \mid \top \mid \bot \mid \neg C \mid (C \sqcap C) \mid (C \sqcup C) \mid \exists R.C \mid \forall R.C,$$

where A is an atomic concept and R is an atomic role. This language is usually denoted $\mathcal{ALC}$. Several sublanguages and extensions of $\mathcal{ALC}$ exist as well.

An example of a concept description is $\mathsf{Woman} \sqcap \forall \mathsf{hasChild}(\mathsf{RedHair} \sqcup \mathsf{BlueEyes})$ describing all women all of whose children either have red hair or blue eyes. To ensure this reading a formal semantics is given to concept descriptions based on the standard set-theoretic semantics of first-order logic, since concepts can be viewed as unary predicates and roles as binary relations. An interpretation $\mathcal{I}$ consists of a non-empty domain $\Delta^{\mathcal{I}}$ and a function that to each atomic concept A assigns a set $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ and to each atomic role R assigns a binary relation $R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. The interpretation function is then extended to all concept descriptions, yielding subsets of the domain $\Delta^{\mathcal{I}}$, in the following inductive way:

$$
\begin{array}{lcl}
\top^{\mathcal{I}} & = & \Delta^{\mathcal{I}} \\
\bot^{\mathcal{I}} & = & \emptyset \\
(\neg C)^{\mathcal{I}} & = & \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}} \\
(C \sqcap D)^{\mathcal{I}} & = & C^{\mathcal{I}} \cap D^{\mathcal{I}} \\
(C \sqcup D)^{\mathcal{I}} & = & C^{\mathcal{I}} \cup D^{\mathcal{I}} \\
(\exists R.C)^{\mathcal{I}} & = & \{a \in \Delta^{\mathcal{I}} \mid \exists b : (a,b) \in R^{\mathcal{I}} \text{ and } b \in C^{\mathcal{I}}\} \\
(\forall R.C)^{\mathcal{I}} & = & \{a \in \Delta^{\mathcal{I}} \mid \forall b : \text{ if } (a,b) \in R^{\mathcal{I}} \text{ then } b \in C^{\mathcal{I}}\}
\end{array}
$$

From the formal semantics it is easy to see that the language $\mathcal{ALC}$ is just a notational variant of standard modal logic, since $C \sqcap D$ corresponds to $C \wedge D$, $C \sqcup D$ corresponds to $C \vee D$, $\exists R.C$ corresponds to $\Diamond_R C$[9], and so on. However, further extensions of $\mathcal{ALC}$ add machinery that makes it into variants of hybrid logic, modal logic with counting quantifiers, or other extended modal logics. For more on the relationship between description logic and modal logic, see [13, 135].

With a formal syntax and semantics for concept descriptions, description logic can now be used to represent knowledge of the world. To do this, another layer is added to description logic, keeping the distinction between terminological knowledge and world assertions in mind, namely TBoxes (terminological boxes) and ABoxes (assertion boxes). A TBox contains terminological knowledge about the world in the form of inclusion axioms $C \sqsubseteq D$ or equality axioms $C \equiv D$ between concept descriptions C and D, for instance $\mathsf{Woman} \sqsubseteq \mathsf{Human}$ or $\mathsf{Human} \equiv \mathsf{Woman} \sqcup \mathsf{Man}$, expressing that all women are humans or that humans are defined as either being a woman or a man. An interpretation $\mathcal{I}$ is said to satisfy a TBox $\mathcal{T}$ if, for all axioms $C \sqsubseteq D$ in $\mathcal{T}$, $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$, and for all axioms $C \equiv D$ in $\mathcal{T}$, $C^{\mathcal{I}} = D^{\mathcal{I}}$.

An ABox contains concrete assertions about the world in the form of concept assertions $C(a)$ or role assertions $R(a,b)$. Here a and b are names of individuals coming from a fixed set of names introduced for expressing world assertions. Thus, an ABox can contain concrete knowledge such as $\mathsf{Woman}(\mathsf{Maria})$ expressing that Maria is a woman, or $\mathsf{hasChild}(\mathsf{Maria}, \mathsf{Peter})$ expressing that Peter is a child of Maria. The notion of an interpretation $\mathcal{I}$ is then extended such that it assigns an object $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$ for each name a. An interpretation

[9]A modal logic can contain several modalities in which case the modalities are indexed as in $\Diamond_R$. Then, in the semantics an accessibility relation R is specified for each modality $\Diamond_R$ and used interpreted formulas $\Diamond_R \varphi$.

$\mathcal{I}$ is then said to satisfy an ABox $\mathcal{A}$ if $a^\mathcal{I} \in C^\mathcal{I}$, for all assertions $C(a)$ in $\mathcal{A}$ and $(a^\mathcal{I}, b^\mathcal{I}) \in R^\mathcal{I}$, for all assertions $R(a, b)$ in $\mathcal{A}$. Note that assertions from ABoxes can be internalized into the syntax of concept descriptions, exactly as hybrid logic internalizes the semantics of modal logic.[10] Thus, the resulting description logic becomes a notational variant of hybrid logic.

With ABoxes and TBoxes, knowledge about the world can be represented in a uniform and concise way. However, description logic offers more than just a smart language for representing knowledge. Due to the formal semantics, TBoxes and ABoxes can contain a great deal of implicit knowledge which can be uncovered by description logic reasoning. Given a TBox $\mathcal{T}$ and a concept description C one can ask whether C is satisfiable with respect to $\mathcal{T}$, that is, whether there is an interpretation $\mathcal{I}$ that satisfies $\mathcal{T}$ and such that $C^\mathcal{I}$ is non-empty. Furthermore, one can ask whether C is subsumed by D with respect to $\mathcal{T}$, that is, whether for all interpretations $\mathcal{I}$ that satisfy $\mathcal{T}$, $C^\mathcal{I} \subseteq D^\mathcal{I}$. Similarly, one can ask whether two concept descriptions are equivalent or whether they are disjoint. However, in several description logics these tasks can be reduced to one another. More advanced reasoning tasks can be obtained by combining other reasoning tasks. For instance one can ask for all the implicit subsumption relationships between concepts that follow from a TBox, and thereby obtain a *classification* of the TBox – a task useful in many applications (such as in medical ontologies) and further discussed in Section 1.3.2.1. Given an ABox and a TBox one can ask whether the ABox contains a possible description of the world, in other words whether it is consistent relative to the given TBox. This question amounts to asking whether there is an interpretation that satisfies both the TBox and the ABox simultaneously. Furthermore, one can ask whether an individual a always satisfies a concept C relative to a TBox and an ABox, or whether the role R is satisfied for individuals a and b, or what the most specific concept (relative to the ordering $\sqsubseteq$) that describes a given individual a is.

Even though many of the reasoning tasks for description logic can be reduced to each other, very efficient reasoning procedures have been developed for the individual tasks in minimal languages. In general, the research and development of description logic have been highly motivated by the wish for efficient implementations. The development of these efficient implementations

[10] Allow for names of individuals to appear as atomic concept and let the interpretation of the name a, when appearing as a concept, be the set $\{a^\mathcal{I}\}$. Then an interpretation satisfies $C(a)$ if and only if the concept $a \sqcap C$ is non-empty and it satisfies $R(a, b)$ if and only if the concept $a \sqcap \exists R.b$ is non-empty.

has made description logic very useful for knowledge representation. For more on description logic and its applications see [12].

In Chapter 7, mainly first-order logic is used to discuss how knowledge representation of regulatory relations can be done. However, the chapter also discusses how description logic can be used, as well as what further research on description logic could be valuable for this kind of knowledge representation – an issue returned to in Section 1.3.2.1.

Description logics are logics tailored to representing, structuring, and reasoning about domain knowledge, which makes them very useful for applications that aim at developing automatic tools for handling large amounts of information. The value of description logics for such applications will be explained further in Section 1.3.2.1. When it comes to reasoning about the knowledge possessed by individuals and knowledge about other individuals' knowledge, a logic that makes explicit references to the individuals and their subjective knowledge is needed. Epistemic logic is a logic that does exactly this.

1.1.4 Epistemic logic

Epistemic logic is essentially merely a subfield of modal logic, dealing with modalities involving knowledge and beliefs.[11] However, epistemic logic is interesting in its own right and has been widely studied. This section discusses exactly how epistemic logic fits in with standard modal logic and its possible world semantics, and how it extends standard modal logic.

A first addition made by epistemic logic to standard modal logic is to include several modalities, one for each agent coming from a fixed, finite set of agents (which will be denoted $\mathbb{A}$ in the following). Therefore, models of epistemic logic do not contain a single accessibility relation, but one for each agent $a \in \mathbb{A}$. The standard modalities of epistemic logic are "agent a knows that" and "agent a believes that", usually represented as K_a and B_a (for all $a \in \mathbb{A}$). However, introducing notions of group knowledge and beliefs, other important modalities become "it is common knowledge among the agents in the group G that" and "it is distributed knowledge among the agents in the group G that" (represented as C_G and D_G), and similar for beliefs. Moreover,

[11]Sometimes the term "epistemic logic" is used only for logics that deal with modalities involving knowledge and the term "doxastic logic" is then used to refer to logics dealing with belief modalities. However, in the rest of this thesis the term "epistemic logic" will be used for all logics dealing with knowledge or beliefs modalities.

some uncommon modalities will also be discussed in Chapter 6, namely "agent a is ignorant about" and "agent a doubts whether".

Epistemic logic, in some form, was already being investigated in ancient and medieval logic [34, 73], but modern epistemic logic was first thoroughly initiated by Hintikka's book "Knowledge and Belief – An Introduction to the Logic of the Two Notions" [96]. Even though it was within philosophy that epistemic logic started, it has been commonly used within computer science since the 1980s [57, 119] and within game theory since the 1990s [11, 19], and the many applications of epistemic logic have helped shape the field ever since.

From the time of Hintikka's work, possible world semantics has been widely accepted as the standard semantics for epistemic logic, and has also motivated many of the applications.[12] The possible world semantics reflects the view that something is known to an agent if it is true in all the alternative situations (possible worlds) that the agent can conceive of as possible. Thus, gaining more knowledge corresponds to eliminating more worlds. This view on knowledge, or information, is referred to as "information as range" by Johan van Benthem [1, 155]. How logic in general deals with information is discussed in more details in Section 1.3.2.

The modalities K_a, B_a, C_G, and D_G are all interpreted in the possible world semantics as the $\Box$ modality of standard modal logic, which for the case of K_a exactly gives rise the view of information as range. Usually, for the modality K_a, further requirements are put on the corresponding accessibility relation R_a in the possible world semantics. The most common requirement is to assume that R_a is an equivalence relation[13] for all agents $a \in \mathbb{A}$. The set of formulas that are valid on the class of frames where all accessibility relations are equivalence relations is called the modal logic $\mathsf{S5}$.[14] There has been considerable philosophical debate about whether $\mathsf{S5}$ is too strong a logic for knowledge, since it makes the agents negatively introspective, that is it validates the axiom $\neg K_a\varphi \to K_a \neg K_a \varphi$ expressing that whatever agent a does not know, a knows that he does not know. Weaker logics like $\mathsf{S4}$(where R_a is only assumed to be reflexive and transitive) have been suggested, but in the many applications in computer science [57] and game theory [11, 160] it has

[12] Other semantics than the possible world semantics are possible for epistemic logic, but in this thesis only possible world semantics is considered for epistemic logic as it appears in chapters 4, 5, and 6.

[13] That R_a is an equivalence relation means that R_a is reflexive ($\forall x(R_a(x,x))$), symmetric ($\forall x \forall y(R_a(x,y) \to R_a(y,x))$) , and transitive ($\forall x \forall y \forall z(R_a(x,y) \wedge R_a(y,z) \to R_a(x,z))$).

[14] Given a class of frames, the set of formulas valid on that class is referred to as a logic.

turned out that the logic S5 best captures the required notion of knowledge.[15] In Chapter 6, S5 is assumed as the logic of knowledge, but in chapters 4 and 5 no assumption is put on the accessibility relation.[16]

When the belief modality B_a is interpreted in possible world semantics it is also as the $\Box$ modality with further requirement on the corresponding accessibility relation. The accessibility relation is usually required to be serial, transitive, and Euclidean[17] giving rise to the logic KD45.[18] Chapter 6 is the only chapter dealing explicitly with beliefs, however, essential to that chapter is how beliefs change under public announcements and the logic KD45 does not work well with public announcements, a matter returned to in Section 1.2.3. Instead, in Chapter 6, the framework of plausibility models will be used.

A plausibility model is a possible world model $\mathcal{M} = \langle W, (\leq_a)_{a\in\mathbb{A}}, V\rangle$, where the accessibility relations $\leq_a$ are assumed to be locally connected, converse well-founded preorders. A relation is *locally connected* if; whenever x and y are related (either $x \leq_a y$ or $y \leq_a x$ holds) and y and z are related, then x and z are also related; a relation on W is *converse well-founded* if; every non-empty subset of W has a maximal element; and a relation is a *preorder* if it is reflexive and transitive. Furthermore, equivalence relations $\sim_a$ on W can be defined by requiring that $w \sim_a v$ if, and only if either $w \leq_a v$ or $v \leq_a w$. The resulting equivalence class $|w|_a = \{v \in W \mid v \sim_a w\}$ is called the information cell of agent a at w. The semantics of the belief modality B_a can then be

[15]Assuming that S5 is the right logic for knowledge corresponds to assuming that the formulas $K_a\varphi \to \varphi$, $K_a\varphi \to K_aK_a\varphi$, and $\neg K_a\varphi \to K_a\neg K_a\varphi$ are valid. Thus S5 is the logic in which it is assumed that: whatever is known to an agent is true, whenever an agent knows something the agent knows this fact, and whenever an agent does not knows something the agent knows this fact. Accepting these properties is one way of arguing for S5. Another way is by assuming that the relation R_a is an epistemic indistinguishability relation for agent a in which it is natural to assume that: a cannot distinguish the actual from itself, if a cannot distinguish the world v from w, the a distinguish w from v either, and if a cannot distinguish v from w and u from v, then a cannot distinguish u from w either.

[16]Due to automatic completeness with respect to pure formulas in hybrid logic, which will be discussed in Section 1.2.2, it is easy to extend the results of Chapter 4 to the case of S4 or S5.

[17]A relation R is serial if $\forall x\exists y(R(x,y))$ and Euclidean if $\forall x\forall y\forall z(R(x,y) \wedge R(x,z) \to R(y,z))$.

[18]When moving from knowledge to belief the requirement that knowledge implies truth ($K_a\varphi \to \varphi$ should be abandoned since a belief in φ does not ensures that φ is actually true – beliefs can be wrong. However beliefs should be consistent in the sense that an agent should never believe φ and $\neg\varphi$ at the same time. The change corresponds to replacing the requirement of reflexivity with the requirement of seriality on the accessibility relation, which again corresponds to moving from the logic S5 to KD45.

defined by:

$$\mathcal{M}, w \models B_a\varphi \quad \text{iff} \quad \text{for all } v \in max_{\leq_a}(|w|_a), \mathcal{M}, v \models \varphi.$$

The intuition behind the fact $w \leq_a v$ is that agent a thinks that the world v is at least as plausible as world w, but a cannot tell which of the two is the case. Furthermore, an agent believes something to be the case if it is true in the worlds that the agent considers most plausible. For more on the plausibility framework see Chapter 6.

The semantics of the common knowledge modality C_G and the distributed knowledge D_G modality is a little more involved. For relations $(R_a)_{a\in\mathbb{A}}$ new relations $\bigcup_{a\in\mathbb{A}} R_a$ and $\bigcap_{a\in\mathbb{A}} R_a$ can be defined as the set theoretic union, or the set theoretic intersection, respectively, of the relations $(R_a)_{a\in\mathbb{A}}$. Furthermore, for a relation R, R^* will denote the reflexive transitive closure of R, that is the smallest relation that extends R and is reflexive and transitive. With these definitions fixed, given a possible world model $\mathcal{M} = \langle W, (R_a)_{a\in\mathbb{A}}, V\rangle$ and a $G \subseteq \mathbb{A}$, the semantics of the common knowledge modality C_G and the distributed knowledge D_G modality can be defined by:

$$\mathcal{M}, w \models C_G\varphi \quad \text{iff} \quad \text{for all } v \in W, \text{ if } (w,v) \in \left(\bigcup_{a\in G} R_a\right)^* \text{ then } \mathcal{M}, v \models \varphi$$

$$\mathcal{M}, w \models D_G\varphi \quad \text{iff} \quad \text{for all } v \in W, \text{ if } (w,v) \in \left(\bigcap_{a\in G} R_a\right) \text{ then } \mathcal{M}, v \models \varphi.$$

That something is common knowledge means that everybody knows it and that everybody knows that everybody knows it... and so on. Thus common knowledge can be viewed as an infinite conjunction. This intuition is captured by the given formal semantics. That something is distributed knowledge has the intuition that, if the agents pull all their knowledge together they will know it. Hence, for distributive knowledge, only the worlds that all the agents consider possible need to be consulted, which leads to the given formal semantics.

For many of the applications of epistemic logic it is not just knowledge and belief that are important, but also how knowledge and beliefs of agents evolve over time or during a process. In the next section one approach to the dynamics of knowledge and beliefs is further discussed, namely dynamic epistemic logic.

1.1.5 Dynamic epistemic logic

Dynamic aspects of knowledge and beliefs have been studied in logic for some time, for instance in belief revision [5, 85] or interpreted systems [78, 57, 125].

However, recently dynamic epistemic logic has emerged as an alternative approach. In interpreted systems the dynamics of knowledge and beliefs are hardwired into the system by specifying all the possible runs of the systems as well as how the states of the system can change due to certain actions. Therefore, the entire system and all its future developments have to be specified from the beginning of a modeling process. Dynamic epistemic logic takes another approach, where only the starting state of the system needs to be specified and the future evolution of the system is completely given by the actions performed. So, instead of modeling a system by specifying all the possible runs, one specifies the possible actions instead. This gives a local view of the dynamics of knowledge and beliefs that has proved quite useful in several applications. The applications of dynamic epistemic logic include: verification of security protocols for communications [164, 50, 51, 95, 4], automated epistemic planning [35, 108], reasoning about quantum computation and information [16, 18], elucidating the foundation of game theory [160, 156, 47], modeling dialogues and communication in linguistics and philosophy of language [113] as well as speech acts [105, 106]. In this section the syntax and semantics of dynamic epistemic logic are formally introduced and several issues relevant for this thesis are elaborated.

Dynamic epistemic logic is a term used to cover extensions of epistemic logic that add dynamic modalities (as in dynamic logic [86, 165]), representing actions with epistemic effects, to model the dynamics of knowledge and beliefs. However, instead of interpreting these dynamic modalities as quantifying over possible worlds within a given model, in dynamic epistemic logic the dynamic modalities quantify over transformations of possible world models. In most versions of dynamic epistemic logic, actions having only epistemic effects are considered. For dynamic epistemic logics that can change the fact of the world as well see [162, 157, 105]. Actions only having epistemic effect will also be referred to as epistemic actions.

The simplest version of dynamic epistemic logic, though still giving rise to numerous applications, is public announcement logic, which goes back to a paper by Plaza in 1989 [128], but was also independently developed in the late 1990s by Gerbrandy and Groeneveld [70, 69]. Public announcement logic adds one type of dynamic modality to standard epistemic logic, namely modalities of (truthful) public announcements. To the syntax of epistemic logic a modality $[\varphi]$ is added for every formula in the language, giving rise to complex formulas of the form $[\varphi]\psi$ having the intuitive reading "after public announcement of φ, ψ is the case". When a truthful announcement of φ takes place it means that no agent any longer considers worlds possible where φ was not true.[19] In the formal semantics this simply corresponds to moving the evaluation of a formula to the submodel consisting only of worlds that make φ true. Formally, for any model $\mathcal{M} = \langle W, (R_a)_{a \in \mathbb{A}}, V \rangle$, any $w \in W$, and any formulas of the language φ and ψ, the following clause is added to the inductive definition of the relation $\models$:

$$\mathcal{M}, w \models [\varphi]\psi \quad \text{iff} \quad \mathcal{M}, w \models \varphi \text{ implies that } \mathcal{M}|_\varphi, w \models \psi,$$

where $\mathcal{M}|_\varphi = \langle W|_\varphi, (R_a|_\varphi)_{a \in \mathbb{A}}, V|_\varphi \rangle$ is the submodel defined by

$$\begin{array}{lcl} W|_\varphi & = & \{w \in W \mid \mathcal{M}, w \models \varphi\} \\ R_a|_\varphi & = & R_a \cap (W|_\varphi \times W|_\varphi)\ , \quad \text{for all } a \in \mathbb{A} \\ V|_\varphi(p) & = & V(p) \cap W|_\varphi\ , \quad \text{for all } p \in \mathsf{PROP}. \end{array}$$

Normally public announcement logic is viewed as an extension of epistemic logic and therefore the underlying accessibility relations are assumed to be equivalence relations. However, a more general approach will be taken in this thesis, and unless particularly mentioned, no particular assumption on the accessibility relation will be made. Nevertheless, all results, except the ones in Chapter 5, are easily extendable to the case where the accessibility relations are equivalence relations.

As already indicated, the modality $[\varphi]$ models the truthful announcement of φ, which is the reason for the antecedent requirement "$\mathcal{M}, w \models \varphi$" in the definition of the semantics of $[\varphi]\psi$. This means that a public announcement $[\varphi]$ is always treated as incoming true information for all agents of the model which result in them all coming to know that φ was the case. Furthermore, this is all common knowledge among the agents in the model, which is the reason for the term "public".

The presented syntax and semantics for public announcements also fit nicely with the plausibility framework introduced in the last section. The just presented semantic for the public announcement operator is used on plausibility models as well. However, in Chapter 6 where the logic of plausibility models is used together with public announcements, other announcements that

[19] A public announcement of φ does not guarantee that φ becomes true. Take for instance the Moore sentence "p is true but a does not know it". After a public announcement of this formula, a does know p and the Moore sentence therefore becomes false. However, the Moore sentence *was* true at the moment of the announcement. This issue leads to a distinction between successful and unsuccessful formulas, see Section 4.7 in [163].

changes the plausibility relation instead of deleting worlds are also introduced. In chapters 4 and 5 public announcement logic will be combined with hybrid logic as introduced in Section 1.1.2.

Expressiveness and succinctness are important issues for public announcement logic. Due to the following validities in public announcement logic

$$[\varphi]\, p \quad \leftrightarrow \quad (\varphi \to p) \tag{1.1}$$

$$[\varphi]\, \neg\psi \quad \leftrightarrow \quad (\varphi \to \neg[\varphi]\psi) \tag{1.2}$$

$$[\varphi]\, (\psi \wedge \chi) \quad \leftrightarrow \quad ([\varphi]\psi \wedge [\varphi]\chi) \tag{1.3}$$

$$[\varphi]\, K_a\psi \quad \leftrightarrow \quad (\varphi \to K_a[\varphi]\psi) \tag{1.4}$$

$$[\varphi]\, [\psi]\chi \quad \leftrightarrow \quad [\varphi \wedge [\varphi]\psi]\chi, \tag{1.5}$$

every formula of public announcement logic can be transformed into an equivalent[20] formula of standard epistemic logic without public announcements. Thus, the addition of public announcement modalities $[\varphi]$ to epistemic logic does not increase the expressive power of the logic in the sense that nothing new can be expressed, [163]. This is also the case if the distributed knowledge modality or any of the hybrid machinery of Section 1.1.2 are added to standard epistemic logic, which is shown in Chapter 4. However, adding public announcement modalities to epistemic logic with common knowledge does increase the expressive power of the logic – one reason for this is briefly discussed in Section 4.4.5 of Chapter 4. Although public announcements do not add to the expressive power of epistemic logic (with the exception of epistemic logic with common knowledge) they do add to the succinctness of epistemic logic. With public announcements, propositions can be expressed much shorter than in standard epistemic logic [115] (at least for the case where no requirements are put on the accessibility relations). The validities (1.1) – (1.5) are usually referred to as *reduction axioms* because they play an important role in the proof theory of public announcement logic. The proof theory of public announcement logic will be presented in Section 1.2.3 and will be the topic of chapters 4 and 5.

Public announcements are just one simple type of epistemic action corresponding to the simple transformation on possible world models of moving to submodels. However, the real power of dynamic epistemic logic is obtained when a wider range of more complex epistemic actions are dealt with

[20]Two formulas φ and ψ are said to be equivalent if for all models $\mathcal{M} = \langle W, (R_a)_{a\in\mathbb{A}}, V\rangle$ and all $w \in W$, $\mathcal{M}, w \models \varphi$ if, and only if $\mathcal{M}, w \models \psi$.

in a uniform manner by transformations on possible world models. The approach that has become the dominant one, initially developed by Baltag, Moss, and Solecki [15], interprets dynamic epistemic modalities as "action models", which, through a product operation, transform possible world models and thereby bring dynamics into epistemic logic.

Action models resemble possible world models and consist of a set of *events*, an accessibility relation between the events for each agent and a precondition function. Formally an action model is a tuple $\mathsf{M} = \langle \mathsf{S}, (\mathsf{Q}_a)_{a \in \mathbb{A}}, \mathsf{pre} \rangle$, where S is the non-empty, finite set[21] of events, $\mathsf{Q}_a \subseteq \mathsf{S} \times \mathsf{S}$ is the accessibility relation for each agent $a \in \mathbb{A}$, and pre is a function that to each $\mathsf{s} \in \mathsf{S}$ assigns a formula of the language, called the precondition of s. A pointed action model is a pair (M, s), where $\mathsf{M} = \langle \mathsf{S}, (\mathsf{Q}_a)_{a \in \mathbb{A}}, \mathsf{pre} \rangle$ is an action model and $\mathsf{s} \in \mathsf{S}$. The intuition behind this definition is that (M, s) represents an epistemic action. When the action (M, s) is preformed, the agents might be uncertain about exactly which event this constitutes, which is represented by the action model M, but the actual event taking place is s. Thus, the set S represents all the events that the agents consider possible with respect to the action, and the relation Q_a represents which events agent a cannot distinguish. Furthermore, each event comes with a precondition that needs to be satisfied for the event to take place.

For every epistemic action (M, s) a modality $[\mathsf{M}, s]$ is added to the syntax giving rise to complex formulas of the form $[\mathsf{M}, s]\varphi$.[22] Semantically, the modality $[\mathsf{M}, \mathsf{s}]$ is interpreted by a product operation between possible world models and actions models. Formally, given a possible world model $\mathcal{M} = \langle W, (R_a)_{a \in \mathbb{A}}, V \rangle$ and an action model $\mathsf{M} = \langle \mathsf{S}, (\mathsf{Q}_a)_{a \in \mathbb{A}}, \mathsf{pre} \rangle$, the product $\mathcal{M} \otimes \mathsf{M} = \langle W', (R'_a)_{a \in \mathbb{A}}, V' \rangle$ is the possible world model defined by:

$$\begin{array}{lll} W' & = & \{(w, \mathsf{s}) \in W \times \mathsf{S} \mid \mathcal{M}, w \models \mathsf{pre}(\mathsf{s})\} \\ R'_a\big((w, \mathsf{s}), (v, \mathsf{t})\big) & \textit{iff} & R_a(w, v) \text{ and } \mathsf{Q_a}(\mathsf{s}, \mathsf{t}), \quad \text{for all } a \in \mathbb{A} \\ V'(p) & = & \{(w, \mathsf{s}) \in W' \mid w \in V(p)\}, \quad \text{for all } p \in \mathsf{PROP}. \end{array}$$

[21] For the specification of the syntax and semantics, the requirement of finiteness is not essential, however, it is essential for the proof theory of dynamic epistemic logic as discussed in Section 1.2.3.

[22] It might appear that semantics, in the form of action models, is introduced into the syntax. In some sense this is true, but it can be avoided by introducing syntactic names for each action, see section 6.1 of [163]. Note that, to avoid self references, the formulas occurring as preconditions in (M, s) need to be constructed before the formula $[\mathsf{M}, s]\varphi$ is constructed. This, however, can easily be ensured by an inductive definition.

The formal semantics of the action modality $[\mathsf{M},\mathsf{s}]$ can now be defined by:

$$\mathcal{M}, w \models [\mathsf{M},\mathsf{s}]\varphi \quad \textit{iff} \quad \mathcal{M}, w \models \mathsf{pre}(\mathsf{s}) \text{ implies that } \mathcal{M} \otimes \mathsf{M}, (w,\mathsf{s}) \models \varphi.$$

The requirement of $\mathcal{M}, w \models \mathsf{pre}(\mathsf{s})$ for (w,s) to be included in W', reflects the idea that the precondition $\mathsf{pre}(\mathsf{s})$ needs to be satisfied in a world before the event s can take place in that world. The definition of R'_a reflects the intuition that an agent can distinguish between two resulting worlds (w,s) and (w',s'), either if the agent can distinguish between the original worlds w and w' or if the agent can distinguish the events s and s' as they occur. Finally, the definition of V' reflects the fact that epistemic actions cannot change the fact of the world, represented by the value of the propositional variables in PROP.

It is worth noticing that dynamic epistemic logic with action models is a genuine generalization of public announcement logic, since public announcements can be described by action models. Given a formula φ, a public announcement of φ corresponds to the action model $\langle\{\mathsf{s}_0\}, \{(\mathsf{s}_0,\mathsf{s}_0)\}, \{(\mathsf{s}_0,\varphi)\}\rangle$. It is easy to see that this action model results in the same model transformations as the public announcement operator $[\varphi]$.

Surprisingly, as in the case of public announcements, adding action modalities of the form $[\mathsf{M},\mathsf{s}]$ to epistemic logic does not increase the expressive power of the language. This is again due to the existence of valid reduction axioms:

$$[\mathsf{M},\mathsf{s}]\, p \quad \leftrightarrow \quad \big(\mathsf{pre}(\mathsf{s}) \rightarrow p\big) \tag{1.6}$$

$$[\mathsf{M},\mathsf{s}]\, \neg\varphi \quad \leftrightarrow \quad \big(\mathsf{pre}(\mathsf{s}) \rightarrow \neg[\mathsf{M},\mathsf{s}]\varphi\big) \tag{1.7}$$

$$[\mathsf{M},\mathsf{s}]\, (\varphi \wedge \psi) \quad \leftrightarrow \quad \big([\mathsf{M},\mathsf{s}]\varphi \wedge [\mathsf{M},\mathsf{s}]\psi\big) \tag{1.8}$$

$$[\mathsf{M},\mathsf{s}]\, K_a\varphi \quad \leftrightarrow \quad \big(\mathsf{pre}(\mathsf{s}) \rightarrow \bigwedge_{R_a(\mathsf{s},\mathsf{t})} K_a[\mathsf{M},\mathsf{t}]\varphi\big) \tag{1.9}$$

$$[\mathsf{M},\mathsf{s}]\, [\mathsf{M}',\mathsf{s}']\varphi \quad \leftrightarrow \quad [(\mathsf{M};\mathsf{M}'),(\mathsf{s},\mathsf{s}')]\varphi, \tag{1.10}$$

where, in the last formula, the ";" operation is a semantic operation on action models: Given two action models, $\mathsf{M} = \langle \mathsf{S}, (\mathsf{Q_a})_{a\in\mathbb{A}}, \mathsf{pre}\rangle$ and $\mathsf{M}' = \langle \mathsf{S}', (\mathsf{Q}'_\mathsf{a})_{a\in\mathbb{A}}, \mathsf{pre}'\rangle$, the composition $(\mathsf{M};\mathsf{M}') = \langle \mathsf{S}'', (\mathsf{Q}''_\mathsf{a})_{a\in\mathbb{A}}, \mathsf{pre}''\rangle$ is defined by:

$$\begin{array}{lcl} \mathsf{S}'' & = & \mathsf{S} \times \mathsf{S}' \\ \mathsf{Q}''_\mathsf{a}\big((\mathsf{s},\mathsf{s}'),(\mathsf{t},\mathsf{t}')\big) & \textit{iff} & \mathsf{Q_a}(\mathsf{s},\mathsf{t}) \text{ and } \mathsf{Q}'_\mathsf{a}(\mathsf{s}',\mathsf{t}') \\ \mathsf{pre}''((\mathsf{s},\mathsf{s}')) & = & \langle \mathsf{M},\mathsf{s}\rangle \mathsf{pre}'(\mathsf{s}'). \end{array}$$

As for public announcement logic, adding action modalities to epistemic logic with common knowledge increases the expressive power. However, whether adding action modalities to a hybrid version of epistemic logic increases the expressive power, is still an open problem. Normally, nominals in hybrid logic are true in exactly one (or at most one) world, but a world w in a possible world model can turn into several worlds (w, s) when a product is taken with an action model. Thus, there seems to be no single obvious way of defining the semantics of nominals in the presence of action models.

Action models will only appear in Chapter 5 where the reduction axioms (1.6) – (1.10) are used as rules for a tableau system for epistemic logic with epistemic actions. As mentioned above, the proof theory of dynamic epistemic logic is discussed in Section 1.2.3, but before turning to proof theory, one final extension of standard modal logic is discussed.

1.1.6 Many-valued logics

Even though modal logic, and the extensions presented here, are often categorized as non-classical logic, they are still classical in the sense that propositions are assumed to be either true or false, and not allowed to be neither or both. This assumption of only two possible truth values is a simplification in several scenarios. Many-valued logic steers clear of this assumption by allowing all sorts of sets to play the role of truth values. However, arbitrary sets are not allowed as truth values, the sets need to have particular structures for the logics to be interesting. Even with this limitation there is still a plenitude of many-valued logics.

For each class of particular structures, a family of many-valued logics arises. In this thesis only one such family will be considered, namely the family arising from requiring that the sets of truth values are finite Heyting algebras. A finite *Heyting algebra* is a finite lattice where every element has a relative pseudo-complement. A *lattice* is a partially ordered set[23] $\mathcal{L} = \langle L, \leq \rangle$ where every two elements $x, y \in L$ always have a least upper bound and a greatest lower bound. For $x, y \in L$ the least upper bound is also called a *join*, denoted by $x \sqcup y$ and the greatest lower bound is also called a *meet*, denoted by $x \sqcap y$.[24] Since only finite Heyting algebras, and thus only finite lattices, are considered, all the lattices will be *bounded* and *complete*. This means that all the lattices $\mathcal{L} = \langle L, \leq, \sqcup, \sqcap \rangle$ will contain a smallest element, denoted $\bot$ and largest element, denoted $\top$, and that for all subsets $A \subseteq L$ the join $\bigsqcup A$ (the least upper bound of the set A) and the meet $\bigsqcap A$ (the greatest lower bound of the set A) can be formed. That

[23]That is, the relation $\leq$ is reflexive, anti-symmetric, and transitive.

[24]The $\sqcap$ and $\sqcup$ is not to be confused with the description logic concept constructors.

all elements of a lattice $\mathcal{L} = \langle L, \leq, \sqcup, \sqcap \rangle$ have a relative pseudo-complement means that, for all a and b in L, there is a greatest element $x \in L$ satisfying $a \sqcap x \leq b$. The element x is called the relative pseudo-complement of a with respect to b and will be denoted by $a \Rightarrow b$. For more on lattices, orders, and Heyting algebras see [48, 33].

The structure of the Heyting algebra allows for natural interpretation of the standard logical connectives; $\wedge$ is interpreted as $\sqcap$, $\vee$ is interpreted as $\sqcup$, $\rightarrow$ is interpreted as $\Rightarrow$, $\top$ is interpreted as the largest element of the Heyting algebra (also denoted $\top$), and $\perp$ is interpreted as the smallest element of the Heyting algebra (also denoted $\perp$).

A possible intuition behind adopting finite Heyting algebras as sets of truth values is given by Fitting in [60, 61, 63], where each truth value is viewed as a subset of a set of experts and the Heyting algebra structure comes from assuming a possible relation of dominance between the agents. Viewing truth values as sets of experts or agents will give way to an application described in Section 1.3.3.

Many-valued logics, where the sets of truth values are finite Heyting algebras, have been combined with modal logic by Fitting in [59, 60, 61]. In his many-valued modal logic, propositions are assigned truth values from a fixed Heyting algebra at each world, and furthermore, the accessibility between worlds is also assigned values from the Heyting algebra.[25] Chapters 2 and 3 concern an extension of this many-valued modal logic to a hybrid logic version.

In Chapter 2 it is shown that the many-valued modal logic of Fitting can be extended to a hybrid version, which shows the great universality of hybrid logic. However, as already mentioned, the semantics of the hybrid machinery in a many-valued setting can be defined in several ways, and even though the way it is defined in Chapter 2 results in a many-valued hybrid logic inheriting many of the properties of standard hybrid logic, there are other natural ways of defining the semantics of the hybrid machinery in a many-valued setting, and this is the topic of Chapter 3.

One of the properties that the many-valued hybrid logic of Chapter 2 inherits from standard hybrid logic is a nice proof theory. The next section discusses proof theory in more detail.

[25] Another reason for assuming the set of truth values to be a finite Heyting algebra is that the notion of relative pseudo-complement allows for a natural way of interpreting the modalities when the accessibility relation is also many-valued. See Chapter 2 for the definition of the semantics of the modalities.

1.2 The logic toolbox II: Proof theory

Proofs have played a role in mathematics since at least ancient times and the rigorous presentation of geometry in Euclid's Elements has been a paradigmatic example of mathematical proofs ever since. Proof theory does not as such study mathematicians' proofs, but rather formal versions of proofs. As a subfield of logic, proof theory grew out of the problems of finding a firm foundation for mathematics, most famously explored by Hilbert and subsequently referred to as "Hilbert's program" [174]. Hilbert realized that a firm foundation for mathematics must rely on logic, something Frege and Russell had already worked on, and in the process he developed an axiomatic proof theory that dominated around 1920 [169]. This style of proof theory is referred to as "Hilbert-style proof systems" or "Hilbert-style axiom systems". Since then several other proof systems have been developed such as natural deduction, sequent calculus (both developed by Gentzen), tableau system (developed by Beth), and resolution (developed by Robinson). In this thesis only Hilbert-style proof systems (Chapter 4) and tableau systems[26] (chapters 2 and 5) are considered. Therefore, the next section gives a short introduction to Hilbert-style proof systems and tableau systems for modal logic, other proof systems for modal logic can be found in [62]. Furthermore, since the logics of chapters 4, 2, and 5 are all hybrid logics, the proof theory of hybrid logic will be discussed in Section 1.2.2, and the proof theory of dynamic epistemic logic will be discussed in Section 1.2.3 since chapters 4 and 5 also contribute to the proof theory of dynamic epistemic logic.

1.2.1 The proof theory of modal logic

While the semantic (or model theoretic) tradition in logic approaches the problem of the validity of arguments through a notion of truth, proof theory approaches the problem through a notion of proofs – an argument is proof-theoretical valid if the conclusion can be proved from the premises. However, within proof theory much effort is put into showing that the proof-theoretical notion of validity coincides with the semantic notion of validity. In the rest of this thesis when referring to a notion of validity it will always be a semantic notion of validity. When a formula is proof-theoretical valid it will usually be referred to as *provable* instead. Thus, the semantic notion of validity will

[26] Sequent calculus and tableau systems are closely connected and for the basic logics they give rise to equivalent proof systems.

be taken as primary, and one of the main task of proof theory will be the task of capturing the semantic notion of validity by a notion of proof.[27] As stated by Fitting in [62], a formal proof is a finitary certificate of the validity of a formula (or an argument). A proof system is something that specifies whether a given object counts as a formal proof of a formula or not. Thus, the objective of proof theory is to develop proof systems that guarantee that every valid formula has a formal proof and that no invalid formula has a formal proof. Moreover, proof theory is a syntactic approach to the validity of arguments, since proof systems specify rules and axioms that lead to formal proofs based on the syntactic structure of formulas without taking the meaning/semantics of the formulas into account.

When developing a proof system for a given logic, as already mentioned, it is important that every valid formula has a formal proof and that no invalid formula has a formal proof. The property that every valid formula has a formal proof is referred to as *completeness* of the proof system. The property that no invalid formula has a formal proof is referred to as *soundness* of the proof system. Besides completeness and soundness, another important property of some proof systems is the question of whether they give rise to a *decision procedure* for the logic. A decision procedure is an algorithm that always terminates and for any formula of the logic tells whether it is valid or not. If such a decision procedure exists, the logic is said to be *decidable*.

In Hilbert-style proof systems, a proof of a formula φ is a finite sequence of formulas where each formula in the sequence is either an axiom (from a fixed set of axioms) or follows by a rule (from a fixed set of rules) from formulas earlier in the sequence, and where the last formula of the sequence is φ. Hence, to specify a Hilbert-style proof system one needs to specify a set of axioms (a particular set of formulas) and a set of rules (a rule takes a fixed finite number of formulas of certain types and returns a single formula). A typical way of giving a Hilbert-style proof system for standard modal logic is shown in Figure 1.1.[28]

[27] This view puts proof theory secondary to semantics, which of course is debatable from a philosophical point of view. The field of proof-theoretic semantics is an example of an approach that takes the notion of proofs prior to a semantic notion of truth.

[28] In several definitions of Hilbert-style proof systems for modal logic an additional rule is included, which states that from a formula φ, any formula φ' obtained from φ by uniformly substituting modal formulas for propositional variables in φ may be inferred. However, such a rule does not preserve validity for dynamic epistemic logic and has therefore been left out in all Hilbert-style proof systems in this thesis. The way the axiom (Tau) is formulated

Axioms:

All substitution instances of propositional tautologies	(Tau)
$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$	(K)

Rules:

From φ and $\varphi \to \psi$ infer ψ	(Modus ponens)
From φ infer $\Box\varphi$	(Necessitation)

Figure 1.1: A Hilbert style proof system for standard modal logic

A few comments are in order. By "substitution instances of propositional tautologies" is meant that whenever φ is a tautology, then φ', obtained by uniformly substituting modal formulas for propositional variables in φ, is also an axiom, for instance $\Box p \vee \neg\Box p$ is an axiom. The axiom (K) is actually an *axiom scheme* (and so is (Tau)), that is, it represents infinitely many axioms, one for every pair of modal formulas φ and ψ. In the following, axiom schemes will also be referred to just as axioms, whenever there is no risk of confusion. A similar issue is the case for the rules; a rule like (Necessitation) is intended to be applicable to any pair of formulas φ and $\varphi \to \psi$. Finally, in this proof system $\Diamond$ is taken to be defined in the syntax as $\neg\Box\neg$, otherwise an axiom of the form $\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$ needs to be included in the proof system.

K will be used to refer to the proof system of Figure 1.1 (K stands for Kripke and **K** refers to the standard modal logic). If there is a proof of the formula φ in **K**, then φ is said to be provable (in **K**), which is written as $\vdash_{\mathbf{K}} \varphi$. It can be shown (see [62, 27]) that the formulas provable in **K** are exactly the formulas valid on the class of all possible world models, i.e. the proof system **K** is sound and complete with respect to standard modal logic. Stated differently **K** is sound and complete with respect to the class of all possible world models.

Soundness for Hilbert-style proof systems amounts to showing that every provable formula is valid. This is done in turn by showing that every axiom is valid and that every rule preserves validity, neither of which are hard to prove. Showing that an axiom is valid or that a rule preserves validity will also be referred to as showing soundness of the axiom or rule.

To prove completeness one has to show that every valid formula is provable,

ensures that the rule is not needed.

Name of proof system:	Additional axioms:	Sound and complete w.r.t. the class of:
T	$\Box\varphi \to \varphi$	reflexive frames
S4	$\Box\varphi \to \varphi$ $\Box\varphi \to \Box\Box\varphi$	reflexive and transitive frames
S5	$\Box\varphi \to \varphi$ $\Box\varphi \to \Box\Box\varphi$ $\neg\Box\varphi \to \Box\neg\Box\varphi$	equivalence relation frames
KD45	$\Box\varphi \to \Diamond\varphi$ $\Box\varphi \to \Box\Box\varphi$ $\neg\Box\varphi \to \Box\neg\Box\varphi$	serial, transitive and Euclidian frames

Figure 1.2: Proof systems for a few modal logics

which is usually done by showing that if a formula is not provable a model can be constructed in which the negation of the formula is true at some world, implying that the formula is not valid. This is usually more complicated than showing soundness, but the standard method is quite clear and will be sketched here: A set of formulas Σ is said to be *consistent* if there is no finite list of formulas $\varphi_1, \varphi_2, ..., \varphi_n$ from Σ such that $\varphi_1 \wedge \varphi_2 \wedge ... \wedge \varphi_n \to \bot$ is provable. A *maximal consistent* set of formulas is a consistent set where no formula can be added without destroying consistency. It can be shown that every consistent set can be extended to a maximal consistent set. Now a *canonical model* can be built by taking the set of possible worlds to be all the maximal consistent sets. Then, it can be shown that a formula belongs to a maximal consistent set if and only if it is true at the maximal consistent set (considered as a world in the canonical modal). Finally if a formula φ is not provable, the set $\{\neg\varphi\}$ is consistent and can thus be extended to a maximal consistent set that in the canonical model satisfies $\neg\varphi$, hence φ is not valid.

The Hilbert-style proof systems retain great flexibility in the sense that new proof systems for other modal logics are easily obtained by simply adding new axioms and in many cases the just-sketched way of showing completeness can be adapted, [27]. For instance adding the axiom scheme $\Box\varphi \to \varphi$ to the proof system of Figure 1.1 will result in a proof system called **T** that is sound and complete with respect to the class of reflexive frames. A few other proof systems are summarized in Figure 1.2.

The last column states, for every proof system, with respect to which logic they are sound and complete, where the logics are defined as the set of valid formulas on the mentioned class of frames. Note that **S5** is used to refer to a proof system and $\mathsf{S5}$ is used to refer to a logic (the set of formulas valid on the class of equivalence relation frames), cf. Section 1.1.4. Sometimes given a proof system, like **S5**, all the provable formulas will also be referred to as a logic. Thus, a logic can both be specified syntactically as the set of provable formulas of a proof system or semantically as the set of formulas valid on a class of frames. In the case of the proof system **S5**, and the other proof systems mentioned, there is no reason for confusion, since the logic specified by the proof system **S5** is equal to the logic $\mathsf{S5}$ – this is exactly what the soundness and completeness of **S5** ensure. Syntactically specified modal logics obtained by adding axioms to the proof system of Figure 1.1 are referred to as *normal modal logics*. Every logic specified as the set of validities on a class of frames is normal, however not every normal modal logic can be obtained as the logic of a class of frames [27].

The possibility of obtaining new proof systems by just adding axioms is an advantage that has been widely used for providing proof systems for dynamic epistemic logic, an advantage used in Chapter 4. How to obtain Hilbert-style proof systems for dynamic epistemic logic is further discussed in Section 1.2.3. There are, however, also disadvantages of Hilbert-style proof systems. The most prominent is the fact that coming up with an actual proof of provable formulas can be quite hard; given a formula, there is no obvious "mechanical" way of finding a proof of it.[29] With the need for automated reasoning, in for instance computer science, this is a great disadvantage of Hilbert-style proof systems. Therefore several other proof systems have been developed that allow for "mechanical" ways of finding proofs. One such is tableau systems (or tableau calculus), which will now be introduced. The issue of automated reasoning is returned to in Section 1.3.2.2.

Whereas Hilbert-style proof systems start with axioms and aim at arriving at the formula needing to be proved, tableau systems start with the formula needing to be proved. Tableau systems prove formulas indirectly by searching for a counter model for the formula in such a systematic way that, if the for-

[29] In Hilbert-style proof systems there is no obvious way of constructing a proof of a formula. A formula φ, not containing a $\Box$ as its main connective, can only be introduced using the modus ponens rule to formulas ψ and $\psi \to \varphi$, however, there is no obvious way of telling what the formula ψ should be.

$$
\begin{array}{c}
F(p \vee q) \to (\neg p \to q) \\
T(p \vee q) \\
F(\neg p \to q) \\
\begin{array}{cc}
Tp & Tq \\
T\neg p & T\neg p \\
Fq & Fq \\
Fp & \mathrm{X} \\
\mathrm{X} &
\end{array}
\end{array}
$$

Figure 1.3: A tableau proof of the formula $(p \vee q) \to (\neg p \to q)$

mula is valid, such a search will show the impossibility of a counter model.[30] This shift makes the process of actually finding proofs much easier and in several cases decision procedures can easily be constructed from tableau systems.

Tableau proofs (or just tableaux) are downward "growing" trees[31] with a formula at each node, and a tableau system identifies the permissible tableau proofs by specifying a set of tableau rules for constructing such trees. Tableau proofs are inductively constructed from the root (usually containing the negation of the formula needing a proof) using rules that specify which new formulas can be added to the end of a given branch of the tree or how a given branch can be split into several new branches. The best way to understand tableau proofs is through an example. Figure 1.3 contains an example of a tableau proof of the formula $(p \vee q) \to (\neg p \to q)$ (according to a tableau system to be specified):

In addition to rules specifying how to inductively construct tableaux, closure rules specify when a tableau branch is called closed and no further rules can be applied to it. (The two branches of the tableau in Figure 1.3 are closed which is indicated by the X's at the end of the branches.) If all the branches of a tableau are closed, the tableau is said to be closed and intuitively it shows that no counter model can be constructed for the formula in question. This much is common for all tableau systems, but otherwise tableau systems come in many versions. Here, a signed tableau system for propositional logic as well as a prefixed tableau system for modal logic will be introduced. In Chap-

[30] Given a formula φ, a counter model for φ is a model that contains a world that satisfies $\neg\varphi$.

[31] What a downward growing tree is will be assumed to be known, otherwise see the formal definition in [143].

$$\frac{T\neg\varphi}{F\varphi}\,(\mathbf{T}\neg) \qquad \frac{F\neg\varphi}{T\varphi}\,(\mathbf{F}\neg) \qquad \frac{T(\varphi\wedge\psi)}{\begin{array}{c}T\varphi\\ T\psi\end{array}}\,(\mathbf{T}\wedge) \qquad \frac{F(\varphi\wedge\psi)}{F\varphi \mid F\psi}\,(\mathbf{F}\wedge)$$

$$\frac{T(\varphi\vee\psi)}{T\varphi \mid T\psi}\,(\mathbf{T}\vee) \qquad \frac{F(\varphi\vee\psi)}{\begin{array}{c}F\varphi\\ F\psi\end{array}}\,(\mathbf{F}\vee) \qquad \frac{T(\varphi\rightarrow\psi)}{F\varphi \mid T\psi}\,(\mathbf{T}\rightarrow) \qquad \frac{F(\varphi\rightarrow\psi)}{\begin{array}{c}T\varphi\\ F\psi\end{array}}\,(\mathbf{F}\rightarrow)$$

Figure 1.4: Signed tableau system for propositional logic.

ter 2 a signed tableau system will be presented for a many-valued hybrid logic and in Chapter 5 a prefixed tableau system will be given for a hybrid public announcement logic as well as a logic containing action models.

In a signed tableau system the formulas occurring on the branches of the tableaux will all be signed formulas of the form $T\varphi$ or $F\varphi$ for formulas φ. The intuition is that $T\varphi$ asserts φ to be true, whereas $F\varphi$ asserts φ to be false. A branch is said to be closed if it contains both $T\varphi$ and $F\varphi$ for some formula φ, or if it contains $T\bot$ or $F\top$. Usually each logical connective gives rise to a rule involving T and a rule involving F. The tableau rules for propositional logic are shown in Figure 1.4 (where the connective $\leftrightarrow$ has been left out).[32]

The reading of a rule like $(\mathbf{T}\wedge)$ is that if $T(\varphi\wedge\psi)$ occurs on a branch, then the two formulas $T\varphi$ and $T\psi$ are both added to the end of the branch. The reading of a rule like $(\mathbf{F}\wedge)$ is that if $F(\varphi\wedge\psi)$ occurs on a branch, then the branch is split into two branches, one ending with the formula $F\varphi$ and the other one ending with the formula $F\psi$. For a given rule, a formula above the line is called a *premise* and a formula below the line is called a *conclusion*. A signed tableau proof for a propositional formula φ is a closed signed tableau starting with $F\varphi$ as the root formula. Recall the example of a tableau proof of the formula $(p\vee q)\rightarrow(\neg p\rightarrow q)$ given in Figure 1.3.

The presented signed tableau system is not easy to extend to also cover various modal logics. When dealing with modal logic, prefixed tableau systems

[32]It is not hard to see that the signs T and F are not really necessary. $T\varphi$ can be replaced by φ and $F\varphi$ by $\neg\varphi$. The rule $(\mathbf{T}\neg)$ can then be omitted and a tableau system for unsigned formulas of propositional logic is obtained. However, the signs make the intuition behind the tableau rules clear and the signs will also be used for clarity when a tableau system for a many-valued hybrid logic is presented in Chapter 2.

are better suited.[33] The prefixed tableau system introduced now comes from [37], but it is fairly standard for modal logic. *Prefixes* are just symbols coming from a fixed countable infinite set Pref, and σ and τ will normally be used for elements of Pref. In prefixed tableaux both *prefixed formulas* $\sigma\varphi$, where φ is a modal formula and $\sigma \in \mathsf{Pref}$, and *accessibility formulas* $\sigma R\tau$, where $\sigma, \tau \in \mathsf{Pref}$, will occur. The intuition behind the prefixes is that they represent possible worlds; $\sigma\varphi$ represents that φ is true in the world σ and $\sigma R\tau$ represents that the world τ is accessible from the world σ.

A branch in a prefixed tableau is said to be closed if it contains both $\sigma\varphi$ and $\sigma\neg\varphi$ for some formula φ (note that the prefix σ is the same in both prefixed formulas). The rules for the prefixed tableau system for standard modal logic are given in Figure 1.5. The reading of the rules is similar to the signed tableau system; however, in the rules ($\Box$) and ($\neg\Diamond$) there are two premises which need to be on the branch before the rules are applicable. (The two premises need not appear in any particular order or be immediately after each other on the branch though.) Furthermore, the rules ($\Diamond$) and ($\neg\Box$) have a side condition stating that the prefix τ must be a new prefix not already occurring on the given branch. A prefixed tableau proof for a modal formula φ is a closed prefixed tableau starting with $\sigma\neg\varphi$ as the root formula. The first tableau system of Chapter 5 is an extension of the tableau system given in Figure 1.5.

Prefixed tableau systems can be extended to other modal logics much easier than tableau systems without prefixes, see [62]. When extending the logic to hybrid logic instead of standard modal logic, tableau systems without prefixes become feasible for a great variety of other modal logics. This issue will be discussed further in Section 1.2.2.

As mentioned earlier, tableau proofs are systematic searches for counter models. Soundness of a tableau system amounts to the fact that all formulas with tableau proofs are valid, i.e., have no counter model. Then, showing that all the tableau rules preserve satisfiability (if all the formulas on a branch are satisfiable then applying any tableau rule will result in at least one new satisfiable branch) ensures soundness: if a formula is not valid, its negation is satisfiable and since all tableau rules preserve satisfiability every tableau starting with the negation of the formula will contain an open branch, and thus the formula does not have a tableau proof (a closed tableau). Showing that a tableau rule preserves satisfiability will also be referred to as showing the

[33] Several prefixed tableau systems for modal logic also use signs such as [62]. However, in this thesis the prefixed tableau systems will be without signs inspired by [37].

$$\frac{\sigma\neg\neg\varphi}{\sigma\varphi}\,(\neg\neg) \qquad \frac{\sigma(\varphi\wedge\psi)}{\begin{array}{c}\sigma\varphi\\ \sigma\psi\end{array}}\,(\wedge) \qquad \frac{\sigma\neg(\varphi\wedge\psi)}{\sigma\neg\varphi \;\big|\; \sigma\neg\psi}\,(\neg\wedge)$$

$$\frac{\sigma(\varphi\vee\psi)}{\sigma\varphi \;\big|\; \sigma\psi}\,(\vee) \qquad \frac{\sigma\neg(\varphi\vee\psi)}{\begin{array}{c}\sigma\neg\varphi\\ \sigma\neg\psi\end{array}}\,(\neg\vee) \qquad \frac{\sigma(\varphi\rightarrow\psi)}{\sigma\neg\varphi \;\big|\; \sigma\psi}\,(\rightarrow) \qquad \frac{\sigma\neg(\varphi\rightarrow\psi)}{\begin{array}{c}\sigma\varphi\\ \sigma\neg\psi\end{array}}\,(\neg\rightarrow)$$

$$\frac{\sigma\Box\varphi \quad \sigma R\tau}{\tau\varphi}\,(\Box) \qquad \frac{\sigma\neg\Box\varphi}{\begin{array}{c}\sigma R\tau\\ \tau\neg\varphi\end{array}}\,(\neg\Box)^1 \qquad \frac{\sigma\Diamond\varphi}{\begin{array}{c}\sigma R\tau\\ \tau\varphi\end{array}}\,(\Diamond)^1 \qquad \frac{\sigma\neg\Diamond\varphi \quad \sigma R\tau}{\tau\neg\varphi}\,(\neg\Diamond)$$

[1] The prefix τ is new to the branch.

Figure 1.5: Tableau system for standard modal logic.

soundness of the rule. Often the tableau rules are constructed in such a way that it is not hard to see that they preserve validity and therefore soundness is something which will not be discussed much for the tableau systems given in chapters 2 and 5.

Completeness amounts to showing that every valid formula has a closed tableau or that every formula for which there exists no closed tableau has a counter model. Here is a sketch of this way of showing completeness. Assume that a formula φ has no closed tableau. The way to prove completeness is then to construct a counter model for φ, which can be done from an open branch of a tableau for φ. For the open branch to be useful it needs to satisfy a certain *saturation property*, which in the prefixed tableau system given for standard modal logic simply amounts to the requirement that no formulas appear more than once on the branch, that the rule $(\Diamond)$ has not been applied to the same formula more than once, and that no more rules are applicable to the formulas on the branch.[34] Completeness of the tableau system of Chapter 2 is basically shown along these lines, but due to the many-valued setting it is a bit more involved. In Chapter 5 completeness of the given tableau systems is only briefly discussed, since it follows more or less straightforwardly from completeness of

[34]For logics where the tableaux are not ensured to be finite, saturation is usually formulated a bit differently.

already existing tableau systems. In fact, completeness of the first tableau system of Chapter 5 follows from the completeness of the tableau system of Figure 1.5. Completeness of the second tableau system of Chapter 5 follows from the completeness of a hybrid tableau system, which will be discussed in Section 1.2.2. The central issue of Chapter 5 is the fact that the given tableau proofs always terminate, something that can lead to a decision procedure for the logics.

As mentioned above, tableau systems are useful for designing decision procedures for logics and thereby showing that they are decidable. The way this is usually done is by showing soundness, completeness, and termination of the tableau system. A tableau system is said to be *terminating* if no infinite tableaux[35] can ever be constructed within the tableau system. How to show that a tableau system is terminating is discussed in more detail in the next section, after having introduced hybrid tableau systems. Before a sound, complete, and terminating tableau system can give rise to a decision procedure, an algorithm needs to be specified, which ensures that tableaux can be constructed in such a way that all branches are either saturated or closed. Then, given a formula of the logic, such a tableau is constructed (which is doable in finitely many steps due to termination). If the tableau is closed the formula is valid (due to soundness), if it contains a branch that is not closed, it is also saturated and the method from the completeness proof shows how a counter model for the formula can be constructed, which means that the formula is not valid. This constitutes a decision procedure for the logic.

1.2.2 The proof theory of hybrid logic

How to extend Hilbert-style proof systems as well as tableau systems to hybrid logic is discussed in this section. The Hilbert-style proof system to be presented is the inspiration for the Hilbert-style proof system for the hybrid public announcement logic given in Chapter 4. The two tableau systems, which will also be presented, are the inspirations for the tableau system for the many-valued hybrid logic given in Chapter 2 and the tableau system for the hybrid public announcement logic given in Chapter 5.

Several Hilbert-style proof systems exist for hybrid logic, but the Hilbert-style proof system of Chapter 4 is based on a proof system from [28] (also presented in [8]). The proof system from [28], for hybrid logic with nominals, satisfaction operators, and the downarrow binder, is shown in Figure 1.6. In

[35] Here a tableau is said to be infinite if it contains infinitely many nodes.

Chapter 4 this proof system is changed somewhat and there are two reasons for this. Since the aim of Chapter 4 is to combine hybrid logic with public announcement logic, the substitution rule (subst) has been left out as it is not valid for public announcement logic.[36] Furthermore, in Chapter 4, nominals are not true in exactly one world but in at most one world, which also forces a change of the proof system.

Axioms:	
All classical tautologies	CL
$\Box(p \to q) \to (\Box p \to \Box q)$	$\mathrm{K}_\Box$
$@_i(p \to q) \to (@_i p \to @_i q)$	$\mathrm{K}_@$
$@_i p \leftrightarrow \neg @_i \neg p$	$\mathrm{Selfdual}_@$
$@_i i$	$\mathrm{Ref}_@$
$@_i @_j p \leftrightarrow @_j p$	Agree
$i \to (p \leftrightarrow @_i p)$	Intro
$\Diamond @_i p \to @_i p$	Back
$@_i(\downarrow s.\varphi \leftrightarrow \varphi[s := i])^1$	DA
Rules:	
From φ and $\varphi \to \psi$, infer ψ	MP
From φ, infer φ^{σ} ^{2}	Subst
From φ, infer $@_i\varphi$	$\mathrm{Gen}_@$
From φ, infer $\Box\varphi$	$\mathrm{Gen}_\Box$
From $@_i\varphi$, where i does not occur in φ, infer φ	Name
From $@_i\Diamond j \to @_j\varphi$ where $i \neq j$ does not occur in φ, infer $@_i\Box\varphi$	BG

^{1} $\varphi[s := i]$ denotes the formula obtained from φ by substituting all free occurrences of s by i. ^{2} φ^{σ} denotes the result of substituting formulas for propositional variables and nominals for nominals according to the substitution σ.

Figure 1.6: The Hilbert-style proof system of [28] for hybrid logic with @ and $\downarrow$.

Soundness of the Hilbert-style proof system is proved in the standard way. Now, however, what is interesting about hybrid logic, and truly makes it a hybrid between modal logic and first-order logic, is the way to show completeness. This is similar to the way completeness is shown for first-order logic. Here is a sketch: Given an unprovable formula φ, a maximal consistent set extending $\{\neg\varphi\}$ is built adding "witnesses" for every modal formula $\Diamond\psi$ in the form of new nominals, just as witnesses in the form of constants are added

[36] In public announcement logic $[p]p$ is valid for propositional variables p, but $[\varphi]\varphi$ is not valid in general.

for existential first-order formulas in the Henkin style completeness proof of first-order logic.[37] Then a model is constructed from the maximal consistent set taking equivalence classes of nominals as worlds (again just like in the Henkin construction for first-order logic). Finally, it is shown that a formula of the form $@_i\psi$ belongs to the maximal consistent set if and only if ψ is true in the model at the world consisting of the equivalence class of i. From this it further follows that the model can be used as a counter model for φ. For an elaboration on this method see Chapter 4.

The new way of showing completeness for hybrid logic has an advantage: it allows for very general completeness results. A *pure formula* is a hybrid logic formula that does not contain any propositional variables. Now, it can be shown that adding any pure formulas as extra axioms to the proof system of Figure 1.6 results in a proof system that is automatically complete with respect to the class of frames the pure formulas define. The class of frames that a set of pure formulas defines is the class of frames on which they are valid. With this general completeness theorem, one obtains completeness for a wide range of hybrid logics since numerous classes can be defined by pure formulas. The way this general completeness result is shown is quite simple. Due to a *frame lemma* it can be shown that, for the model defined in the completeness proof, the underlying frame always validates all pure formulas that have been added as axioms, and thus the model constructed is always a counter model of the "right" kind. In the presence of public announcement modalities the general completeness results also retain, which is one of the main results shown in Chapter 4.

The Hilbert-style proof system for standard modal logic is easily extendable to other modal logics in many cases. This is generally not the case for other proof systems for modal logic without introducing some extra machinery such as prefixes, as in the last tableau system of the previous section. However, for hybrid logic, extra machinery, such as prefixes, is not needed to obtain nice and uniform proof systems such as tableau systems, natural deduction systems, or sequent calculus. This is essentially due to the internalization of the possible world semantics in hybrid logic. For more on the proof theory of hybrid logic and why it is well-behaved, see the recent book [42].

An internalized tableau system for hybrid logic is a tableau system that

[37] That nominals can be used as witnesses is not that surprising, once one is viewing hybrid logic as a fragment of first-order logic and realizes that nominals correspond to constants of first-order logic, see [8].

does not use prefixes, but where prefixes are "simulated" using the satisfaction operators of hybrid logic. This is done by using formulas of the form $@_i\varphi$ instead of $\sigma\varphi$ and $@_i\Diamond j$ instead of $\sigma R\tau$. In this thesis both prefixed and internalized tableau systems are considered for hybrid logic. The prefixed tableau systems for hybrid logic are usually simpler and easier to work with, on the other hand the internalized tableau systems allow for general completeness in the same style as for Hilbert-style proof systems for hybrid logic. In [25] an internalized tableau system is presented and it is shown that it allows for general completeness by adding the following rule: If $\mathcal{H}$ is a set pure formulas, then any of them are allowed to be added to any tableau branch at any time, and so are any pure formulas obtained by uniformly substituting nominals for nominals in formulas of $\mathcal{H}$. [25] then shows that the tableau system with this additional rule is complete with respect to the class of frames defined by the formulas in $\mathcal{H}$. However, when extending tableau systems with pure formulas like this, the tableau system is no longer guaranteed to be terminating.

Terminating tableau systems for hybrid logic have been developed in several versions by Blackburn, Bolander, and Braüner in [38, 39, 37, 36]. The prefixed tableau system of [37] for hybrid logic with nominals, satisfaction operators, and the global modality is shown in Figure 1.7 and is the one extended to hybrid public announcement logic in Chapter 5. However, since nominals are true in at most one world instead of exactly one world in the hybrid public announcement logic of Chapter 5, the rule $(\neg)$ is omitted – which turns out to be the only change needed. As presented above, the tableau system of Figure 1.7 is not terminating. An additional *loop check condition* is needed that blocks the application of the rules $(\neg)$, $(\Diamond)$, $(@)$, $(\neg@)$, and (E) based on properties of the prefix σ. For an elaboration of this see Chapter 5 or [37].

In Chapter 2, prefixes are internalized in the tableau system for many-valued hybrid logic inspired by the tableau system of [39] shown in Figure 1.8. However, due to the many-valued setting of Chapter 2 further rules are needed and the resulting tableau system looks considerably different from the one in Figure 1.8. Furthermore, signs T and F, as in the tableau system of Figure 1.4, are also used to simplify the reading of the rules in Chapter 2. Fitting already gave a signed non-prefixed tableau system in [61] for a many-valued modal logic. Nevertheless, the great advantage of developing an internalized tableau for the many-valued hybrid logic is that it can be shown to terminate. It is not obvious at all how to prove termination of Fitting's tableau system, if it is terminating at all.

Soundness and completeness of the hybrid tableau systems, internalized

$$\frac{\sigma\neg i}{\tau i}\,(\neg)^1 \qquad \frac{\sigma\neg\neg\varphi}{\sigma\varphi}\,(\neg\neg) \qquad \frac{\sigma\varphi\wedge\psi}{\begin{array}{c}\sigma\varphi\\ \sigma\psi\end{array}}\,(\wedge) \qquad \frac{\sigma\neg(\varphi\wedge\psi)}{\sigma\neg\varphi \mid \sigma\neg\psi}\,(\neg\wedge)$$

$$\frac{\sigma\Diamond\varphi}{\begin{array}{c}\sigma R\tau\\ \tau\varphi\end{array}}\,(\Diamond)^1 \qquad \frac{\sigma\neg\Diamond\varphi \quad \sigma R\tau}{\tau\neg\varphi}\,(\neg\Diamond) \qquad \frac{\sigma E\varphi}{\tau\varphi}\,(E)^1 \qquad \frac{\sigma\neg E\varphi}{\tau\neg\varphi}\,(\neg E)^2$$

$$\frac{\sigma @_i\varphi}{\begin{array}{c}\tau i\\ \tau\varphi\end{array}}\,(@)^1 \qquad \frac{\sigma\neg @_i\varphi}{\begin{array}{c}\tau i\\ \tau\neg\varphi\end{array}}\,(\neg @)^1 \qquad \frac{\sigma\varphi \quad \sigma i \quad \tau i}{\tau\varphi}\,(Id)$$

[1] The prefix τ is new to the branch. [2] The prefix τ is already on the branch.

Figure 1.7: The prefixed hybrid tableau system of [37].

$$\frac{@_i\neg\varphi}{\neg @_i\varphi}\,(\neg) \qquad \frac{\neg @_i\neg\varphi}{@_i\varphi}\,(\neg\neg) \qquad \frac{@_i\varphi\wedge\psi}{\begin{array}{c}@_i\varphi\\ @_i\psi\end{array}}\,(\wedge) \qquad \frac{\neg @_i(\varphi\wedge\psi)}{\neg @_i\varphi \mid \neg @_i\psi}\,(\neg\wedge)$$

$$\frac{@_i\Diamond\varphi}{\begin{array}{c}@_i\Diamond j\\ @_j\varphi\end{array}}\,(\Diamond)^{1\ 2} \qquad \frac{\neg @_i\Diamond\varphi \quad @_i\Diamond j}{\neg @_j\varphi}\,(\neg\Diamond) \qquad \frac{@_i @_j\varphi}{@_j\varphi}\,(@) \qquad \frac{\neg @_i @_j\varphi}{\neg @_j\varphi}\,(\neg @)$$

$$\frac{}{@_i i}\,\textbf{(ref)}^3 \qquad \frac{@_i\Diamond j \qquad @_j k}{@_i\Diamond k}\,\textbf{(Bridge)}$$

$$\frac{@_i j \qquad @_i\varphi}{@_j\varphi}\,\textbf{(Nom1)}^4 \qquad \frac{@_i j \qquad @_j\Diamond k}{@_i\Diamond k}\,\textbf{(Nom2)}$$

[1] The nominal j is new to the branch. [2] The formula φ is not a nominal. [3] The nominal i is on the branch. [4] The formula φ is a propositional variable or a nominal.

Figure 1.8: The internalized hybrid tableau system of [39].

or not, are quite similar to soundness and completeness of prefixed tableau systems. Instead of building the counter model based on prefixes, as in the completeness for prefixed tableau systems, one builds the counter model based on the nominals occurring on an open saturated branch, see [37].[38] As already mentioned the many-valued setting of Chapter 2 complicates matters and the completeness proof presented therein deviates considerably from standard completeness proofs for hybrid tableau systems. An open saturated tableau branch does not give rise to a unique counter model anymore, since the tableau branch only provides lower and upper bounds for truth values. Nonetheless, it can be shown that any model build from the branch that respects these bounds, suffices as a counter model.

Termination of tableau systems has been mentioned as an important property several times. In the papers [38, 39], Bolander and Braüner developed a general way of showing termination of tableau systems. It is based on a non-trivial extension of the idea of showing termination for ordinary tableau systems for propositional logic. Here it is observed that application of a rule to a premise of maximal length on a branch always strictly decreases the maximal length of formulas on the branch, and thus the maximal length of the formulas on the branch strictly decreases down along a branch as a rule is applied. Therefore there can be no infinite branches (since the root formula has a fixed finite length) and since the tableau is finitely branching it is necessarily finite. In termination for propositional logic it is also used that no rule can be applied more than once to a formula, however in modal or hybrid logic the rule $(\neg\Diamond)$ can be used several times due to new prefixes (or nominals) being introduced on a branch. Instead, one can show that the maximal length of formulas strictly decreases when moving along the tree constructed from the new prefixes (or nominals) that are introduced on a branch. In addition it just needs to be shown that a branch of a tableau is infinite if, and only if, the corresponding tree of new prefixes has an infinite branch. This method is exemplified nicely in Chapter 2, otherwise see [37]. The method turns out to be quite powerful and general, which is shown by this thesis, where the method is extended both to a many-valued hybrid logic (in Chapter 2) and dynamic epistemic logics (in Chapter 5).

Exploring the proof theory of hybrid logic has been one of the aims of

[38]Due to the inclusion of loop check conditions to ensure termination, branches are not necessarily saturated for all nominals on the branch, but only for a certain kind of nominals, usually referred to as *urfarthers*. However, building the counter model from urfarthers alone suffices for the completeness proof in these cases, see [37].

this thesis; another related aim has been to use the advantages of hybrid logic proof theory to investigate the proof theory of public announcement logic and dynamic epistemic logic in general. Therefore, it is time to look closer at the proof theory of dynamic epistemic logic.

1.2.3 The proof theory of dynamic epistemic logic

Due to the relatively young age of dynamic epistemic logic and the fact that it has been mostly semantically driven, the proof theory of dynamic epistemic logic is somewhat underdeveloped. Furthermore, almost all of the proof systems for dynamic epistemic logic are Hilbert-style proof systems with the exception of [14, 49, 117]. This thesis contributes to the proof theory of dynamic epistemic logic by giving a Hilbert-style proof system for a hybrid public announcement logic that allows for automatic completeness with pure formulas in Chapter 4, and by giving two terminating tableau systems in Chapter 5, one for epistemic logic with action modalities and one for the hybrid public announcement logic of Chapter 4. Therefore this section briefly introduces the standard Hilbert-style proof systems for dynamic epistemic logic and describes how completeness can be shown by reduction axioms.

The standard Hilbert-style proof system for public announcement logic (without common knowledge) is particularly simple, it is obtained by just adding the reduction axioms (1.1) – (1.5) of Section 1.1.5[39] to the Hilbert-style proof system in Figure 1.1 of Section 1.2.1. (If one wants the accessibility relations to be equivalence relations, one just adds the axioms of **S5** for each agent $a \in \mathbb{A}$ to this proof system.) Since (1.1) – (1.5) are valid in public announcement logic, as already mentioned, the resulting proof system is sound.

Completeness of public announcement logic is also easy to obtain, as the following sketch of the proof shows: First, a translation from public announcement logic to standard modal logic (or epistemic logic) is inductively defined as in Figure 1.9. It is then shown that for all formulas φ of public announcement logic $\varphi \leftrightarrow t(\varphi)$ is provable, from which it also follows by the assistance of the soundness of the proof system that $\varphi \leftrightarrow t(\varphi)$ is valid. Thus, if a formula φ is valid in public announcement logic, $t(\varphi)$ is also valid. Furthermore, since the models of public announcement logic and the standard modal logic are the same, $t(\varphi)$ is also valid in standard modal logic, since $t(\varphi)$ is a formula of

[39] Note that the reduction axioms (1.2) – (1.5) are axiom schemes, i.e. an axiom is added for every formula φ, ψ and χ, whereas (1.1) is not an axiom scheme, as only propositional variables are allowed to be substituted for p in (1.1).

$$
\begin{array}{llllll}
t(p) & = & p & t([\varphi]p) & = & t(\varphi \to p) \\
t(\neg\varphi) & = & \neg t(\varphi) & t([\varphi]\neg\psi) & = & t(\varphi \to \neg[\varphi]\psi) \\
t(\varphi \wedge \psi) & = & t(\varphi) \wedge t(\psi) & t([\varphi]\psi \wedge \chi) & = & t([\varphi]\psi \wedge [\varphi]\chi) \\
t(K_a\varphi) & = & K_a t(\varphi) & t([\varphi]K_a\psi) & = & t(\varphi \to K_a[\varphi]\psi) \\
 & & & t([\varphi][\psi]\chi) & = & t([\varphi \wedge [\varphi]\psi]\chi)
\end{array}
$$

Figure 1.9: The translation t from public announcement logic to epistemic logic.

standard modal logic. But then, $t(\varphi)$ is provable by the completeness of standard modal logic. However, because the proof system of public announcement logic is an extension of the one for standard modal logic, $t(\varphi)$ is provable in public announcement logic as well, by which it follows by the provability of $\varphi \leftrightarrow t(\varphi)$ that φ is provable in public announcement logic.

Proving that $\varphi \leftrightarrow t(\varphi)$ is provable in the proof system is a simple induction proof. There is one small catch though. The formula complexity does not decrease every time one step of the translation is performed, and therefore the induction proof cannot be on the formula complexity as normal. However, another complexity measure can be defined that strictly decreases for every step of the translation. The definition of this complexity measure c is shown below and is taken from [163], where much more on completeness for public announcement logic is also discussed.

$$
\begin{array}{lll}
c(p) & = & 1 \\
c(\neg\varphi) & = & 1 + c(\varphi) \\
c(\varphi \wedge \psi) & = & 1 + \max(c(\varphi), c(\psi)) \\
c(K_a\varphi) & = & 1 + c(\varphi) \\
c([\varphi]\psi) & = & (4 + c(\varphi)) \cdot c(\psi)
\end{array}
$$

The Hilbert-style proof system for public announcement logic does not always behave that nicely. First of all, the completeness proof and axiomatisation just sketched above does not work if common knowledge is added, simply because no reduction axiom can exist for the combination of the public announcement modality and the common knowledge modality, see [163]. Therefore common knowledge is left out in chapters 4 and 5. Secondly, the Hilbert-style proof system for public announcement logic is not easy to extend for other frame classes than the class of all frames and the class of equivalence relation frames. For instance, there is a problem with the class of serial frames (and thereby also the class of frames for KD45) because when moving to a submodel, as the public announcement modality $[\varphi]$ does, a serial accessibility relation might lose its seriality. This implies that the meaning of a KD45 belief modality B is different, depending on whether it appears within the scope of a $[\varphi]$ modality or not.[40] This is one of the main reasons that plausibility models are used when modeling beliefs and public announcement, which explains the choice of logic in Chapter 6.

A problem of the same kind occurs when one wants to add nominals to public announcement logic. The semantics of a nominal specifies that it is true in exactly one world, but if it occurs within the scope of a $[\varphi]$ modality, that world might have been removed. Fortunately there is a way around this by letting nominals be true in at most one world both inside and outside the scope of a $[\varphi]$ modality – this is the approach taken in Chapter 4.

When moving from public announcement logic to dynamic epistemic logic with action models (and still staying clear of common knowledge) the same way of defining a Hilbert-style proof system is possible, the reduction axioms (1.6)–(1.10) of Section 1.1.5 are just used instead of the one for public announcement logic. Completeness and soundness are shown in the same way, a new and little more complicated complexity measure is needed though. The details are skipped, but can be found in [163].

When it comes to tableau systems for dynamic epistemic logic, only two such systems already exist, namely [14, 49], which are both tableau systems for public announcement logic. These tableau systems are not based on reduction axioms, but are based on the semantics of the public announcement modality instead. This allows for the tableau system of [14] to be optimal in terms of complexity, although it also complicates the rules. Furthermore, due to the heavy reliance on the specific semantics of the public announcement operator, there is no obvious way of extending the tableau system of [14] to include action modalities.

In Chapter 5, tableau systems are given for a dynamic epistemic logic with action modalities as well as a hybrid public announcement logic. The tableau systems have very simple rules and are obtained by adding reduction axioms

[40] This problem is not as much a problem of the proof theory as it is a problem of the semantics. However, the consequence is that it is not possible to give a sound and complete proof system for a public announcement logic where the modality B is always interpreted as a KD45 modality.

as rules to the already presented tableau systems of figures 1.5 and 1.7 with the need of no further rules. It is thereby shown that reduction axioms are useful in proof theory beyond merely Hilbert-style proof systems. Furthermore, it allows for the reuse of the soundness and completeness of the tableau systems of figures 1.5 and 1.7. Finally, the method for proving termination sketched in the last section can be extended to prove termination of the tableau systems of Chapter 5 using the new complexity measures.

This concludes a general presentation of the toolbox that the first four chapters of this thesis extend on. As mentioned, the toolbox can be used for modeling information, knowledge and beliefs, which will be the subject of the next sections.

1.3 The toolbox at work: modeling information, knowledge, and beliefs

After a thorough inventory of the toolbox it is time to put the tools to work. The tools of the last section have many applications, but here the focus will be on how they are useful in modeling *information*, *knowledge*, and *beliefs*. These concepts are widely used in our everyday life as well as in many sciences, resulting in various views on what the concepts mean. However, the focus here will be on how information, knowledge, and beliefs are viewed and modeled within philosophy and computer science, more specifically within the subfields of *formal and social epistemology* and *multi-agent systems*. Therefore an introduction to what information, knowledge, an beliefs signify in these fields will be given in Section 1.3.1. Within these fields logic has turned out to be a useful tool in modeling and clarifying the concepts. Why this is the case is discussed in Section 1.3.2, where the topics of the chapters 6 and 7 are also put in a greater context. Finally, in Section 1.3.3 a possible future application of the logic introduced in Chapter 2 is discussed.

1.3.1 Information, knowledge, and beliefs in philosophy and computer science

Information is an important part of our life. From bombing the right troops to catching a train to work, the proper information is central to making the right decisions. In our increasingly specialized and fast-moving society some decisions require increased amounts of expert knowledge and others require great coordination among several agents - both humans, computer programs, and robots. This only makes information and how we handle it even more important; so important that we talk about "the information age".

Moreover, the amount of information we produce and store as a human society is vastly increasing due to modern technology. The telecommunication and IT industry in Denmark is estimated to have made 550,000,000,000 registrations of phone calls and internet traffic last year (due to the Danish anti-terror law) [171], and the laws regarding unemployment benefits in Denmark now comprise 22,575 pages of rules [120]. It is estimated that humanity had accumulated 12 exabytes of information in 2003 and this passed 988 exabytes in 2010, which makes the philosopher Luciano Floridi talk about the "zettabyte era"[41] [66].

[41] 1 exabyte equals 10^{18} bytes and 1000 exabytes equals 1 zettabyte.

It seems obvious that information is important, while at the same time providing the human race with great challenges. However, what does information really amount to? Compared to the long history of philosophy it is only recently that information has received profound attention within the field, actually, philosophy of information has now become an independent subfield, [2, 65, 67]. On the other hand, what has received great attention in philosophy since ancient times, is the concept of *knowledge*. In decision-making the right information is vital, but it is only useful if the acting agents possess the information and they trust it enough to act upon it. It is not as much information we base our decisions on, as it is our knowledge.[42]

1.3.1.1 Epistemology

Epistemology is the subfield of philosophy mainly dealing with knowledge, especially the question of what it amounts to for an agent to have knowledge of a proposition[43]. Traditionally, going back to Plato, knowledge has been defined as *justified true belief*. For an agent to know that φ, φ needs to be true, the agent must believe φ, and the agent must be justified in believing φ. However, this view was shaken by Gettier's short paper [71] in 1963 that presented two counter examples to the "justified true belief" analysis of knowledge, which showed that justified true belief in a formula φ can be ascribed to a person without the person having knowledge of φ.

Conventionally, epistemology has mainly focused on analyzing what knowledge or justification amount to, avoiding the Gettier examples, or argued for or against the mere possibility of knowledge. The role of knowledge in decision-making has been left for other disciplines, though, some philosophers, such as Hintikka ([97], chapter 1), have noted the strong ties between knowledge and decision-making. Furthermore, the precise relationship between knowledge and information is something which is open for discussion.[44] The reason

[42] This is of course open to philosophical debate, something which will be ignored here to keep the introduction focused.

[43] Knowledge in the form of "knowing how to..." has received little attention within mainstream epistemology.

[44] According to Luciano Floridi [66] the general definition of information states that information is meaningful, well-formed data and that semantic information is well-formed, meaningful, and truthful data. He further claims that information can give way to knowledge, in the sense that knowledge relates and justifies semantic information [66]. However, the relation to the traditional analysis of knowledge remains vague. Furthermore, this is just one view on the relation between knowledge and information, and several others are possible – once again, a detailed philosophical debate that will lead this introduction astray.

for bringing in epistemology is that it clearly relates to epistemic logic (as introduced in sections 1.1.4 and 1.1.5 and being the concern of chapters 4, 5, and 6). Even though Hintikka introduced epistemic logic into epistemology in 1962 [96], epistemic logic and mainstream epistemology have lived separate lives [90]. This, however, is now changing mainly due to the new subfields of *formal epistemology* and *social epistemology*.

1.3.1.2 Formal epistemology

Broadly viewed, epistemology now deals with knowledge, belief (-change), certainty, rationality, reasoning, decision, justification, learning, agent interaction, and information processing. On such a view, formal epistemology considers the same topics as mainstream epistemology, but attacks them with formal mathematical tools such as logic, probability theory, game theory, decision theory, formal learning theory, belief revision, or distributed computing [89]. There is no doubt though that formal epistemology and mainstream epistemology have focused on quite different aspects of knowledge. For instance, in belief revision and dynamic epistemic logic, the focus in not on an adequate definition of knowledge as in mainstream epistemology, but on how knowledge and information change (or should change) as a result of new incoming information or changes in the world. However, there is nothing to hinder the two approaches getting together and learning from each other. Viewing mainstream and formal epistemology as being on the same quest is advocated by Vincent Hendricks in [90], for instance. Even though logic is just one of many tools put to work by formal epistemology, there is no doubt that it will continue to be a valuable part of the formal epistemology toolbox. Especially epistemic logic now plays a role in formal epistemology, see for instance [90, 93, 149].

There are several trends in the recent development of epistemic logic that has brought it closer to philosophy. For one, *the dynamic turn* in epistemic logic, most strongly advocated by Johan van Benthem [153], has revolutionized epistemic logic and among other things resulted in the many applications mentioned in Section 1.1.5. The dynamic turn in epistemic logic has emphasized the issue of how knowledge changes, or more generally of how information flows under actions such as observation, inference, or communication [152]. In [150] Johan van Benthem even claims that "...information cannot be understood in isolation from the processes which convey and transform it. No *information without transformation*."

The new logics developed under the dynamic turn possess increased expressive powers to formalize discussions within mainstream epistemology. For instance, several versions of the view that knowledge is belief that is stable under belief revision with any new evidence, have been formalized and compared by Alexandru Baltag and Sonja Smets in [17]. The epistemic logics introduced in chapters 4 and 5 of this thesis are dynamic epistemic logics, and moreover the logical framework of [17] is used in Chapter 6 to formalize the phenomenon of *pluralistic ignorance* and thus, this thesis embraces the dynamic turn.

Pluralistic ignorance is a phenomenon from social psychology, where a group of people does not believe something to be the case, but believe that everyone else in the group believes it to be the case. The phenomenon is explained in much more detail in Chapter 6 and is also briefly discussed in Section 1.3.2.3. One essential feature of the phenomenon is that it is a genuine social phenomenon – no single person and his beliefs can constitute pluralistic ignorance. To capture the fact that the phenomenon involves more than one person, logics incorporating several people or agents are required. The interaction between several agents has also turned out to provide a valuable model of computation leading to the field of multi-agent systems. However, before saying more about multi-agent systems, it is worth elaborating how the "many agents" view has also emerged in epistemology.

1.3.1.3 Social epistemology

The focus on knowledge in the context of several individuals has led to the branch of philosophy called *social epistemology*. Goldman, in [75], distinguishes between three different ways that epistemology can be "social". The first way runs close to mainstream epistemology and acknowledges the fact that epistemic agents are often present in a social reality that affects the way the agents may gain and handle knowledge. For instance, we repeatedly gain knowledge through testimony from others in our everyday life. A typical question here would be whether beliefs obtained through testimony are justified, a question that fits well with the practice of mainstream epistemology.

Another way epistemology could be social is by going beyond individual agents and considering collective agents, such as juries, agencies, corporations etc. (Such collective agents contain individual agents though.) There are several examples from daily life where we ascribe knowledge and beliefs to such collective agents, for instance: "the CIA knew where Osama bin Laden was hiding", "the jury believes that the accused is guilty". The question of what a collective agent may come to know or believe has received increased attention lately and has fostered the study of *judgment aggregation*. Judgment aggregation mainly uses formal tools to study how a collective agent can form a judgment based on the judgments of the individual agents it consists of. Judgment aggregation will be discussed in more detail in Section 1.3.3 since it provides a possible future application of the logic introduced in Chapter 2.

The third way that epistemology can be "social" is through the study of what Goldman [75] calls *epistemic systems*. Collective agents can be epistemic systems, but epistemic systems go beyond mere collective agents. Goldman mentions science, education, and journalism as examples of epistemic systems. For more on this system-oriented social epistemology see [75].

The study of a phenomenon such as pluralistic ignorance can be regarded as social epistemology to the extent that the phenomenon can be viewed as an informational phenomenon. If this is the case, the study will involve aspects of the first two versions of social epistemology. First of all, pluralistic ignorance can be seen as a phenomenon where a collective agent holds a belief that conflicts with the individual agents. Secondly, exactly how information flows between different agents in a case of pluralistic ignorance and affects the agents' individual beliefs is something that is certainly worth studying.[45]

One of the key features of pluralistic ignorance is the fact that people are holding <u>wrong</u> beliefs about other people's beliefs. In the classical account, one thing that distinguishes beliefs from knowledge is the fact that beliefs can be wrong, whereas knowledge cannot. Thus, a logic-based model of pluralistic ignorance requires focus on beliefs instead of knowledge, and it therefore provides a nice example of why it is sometimes useful to shift focus from knowledge to beliefs. What exactly beliefs are is of course also open to philosophical discussion, see [137]. As for knowledge, belief is viewed as a *propositional attitude*, that is, it is a mental state of having some attitude, stance, or opinion towards a given proposition [137]. Viewing knowledge and beliefs this way makes it natural to approach epistemic or doxastic logic in a modal logic way, as in Section 1.1.4. For an overview of other formal approaches to modeling beliefs see [101]. Here it will just be noted that in the usual logic-based modeling of beliefs, the key feature that distinguishes it from knowledge is the lack of veridicality, that is a belief in φ does not imply the truth of φ as is the case for knowledge. However, beliefs are required to be consistent, that is an agent cannot both believe φ and $\neg\varphi$ for some formula φ. In terms of Kripke semantics, the accessibility relation underlying the belief modality is required

[45] In fact, pluralistic ignorance in one of the four central topics in the ongoing series of workshops in social epistemology between Copenhagen and Lund [92].

to be serial instead of the stronger requirement of being reflexive, as required for knowledge.

1.3.1.4 Multi-agent systems

The advancement of computer science and the internet has made a vast amount of information easily available to us, but it has also contributed to a huge expansion in the total amount of information. The increased amount of information makes it still harder to find the right information. Furthermore, we need to communicate the information to a greater extent across several different contexts (patients' information needs to be available to patients, doctors, nurses, lawyers, employers, and public services). Thus, tools that can help us find, understand, manage, and communicate information automatically on larger scales are an enormous advantage. The great increase in information is made possible due to the development of computers, and the majority of computer scientists are focused on handling this information challenge. The field *multi-agent systems* has grown out of computer science and deals with information, knowledge and beliefs in several ways.

Multi-agent systems is a relatively new and fast growing subfield of computer science that approaches the increased complexity and size of computer systems in a novel way. In the multi-agent paradigm, computer systems are assumed to be composed of autonomous agents that constantly make decisions on which actions to take in pursuit of their individual goals within an environment. In deciding which actions to adopt, the agents need to consult their propositional attitudes such as knowledge, beliefs, preferences, and intentions. Furthermore, since the agents often have to communicate, cooperate, coordinate, and negotiate with each other, they also need to reason about other agents' propositional attitudes. Thus, it seems obvious that the field of multi-agent systems has many overlaps with philosophy, especially social epistemology and logic, as well as economics, especially game theory. In general, the multi-agent view on computing is inspired by other fields, such as logic, philosophy, economics, psychology, sociology, and linguistics that use similar agent-based models. Thus, interest in systems of interacting agents spans over a greater scientific community.

There is no unified definition of what a multi-agent system is, however, there are several key features (for more see [172]):

- Multi-agent systems are made up of autonomous agents. This is in contrast to agents that are told what to do in every situation and thus in a multi-agent system there is no central agent in control of the system.

- Agents may be equipped with goals and preferences that guide their decisions and allow them to act on behalf of their creator or whoever owns them.
- Agents are not omnipotent and are not situated in a static environment. Agents are situated in a dynamic environment with other agents with whom they can interact through communication, cooperation, coordination, and negotiation and thereby increase their chances of achieving their goals.

The first two features reflect the fact that agents in a multi-agent system behave *intelligently*, which is the reason why the field of multi-agent systems is closely related to the field of *artificial intelligence*, which goes back to McCarty's in the 1950s [147]. However, one major feature that distinguishes multi-agent systems from artificial intelligence is the inclusion of the third feature mentioned above. A key feature of multi-agent systems is that the agents are situated in an environment of other agents, or put differently that they are always members of an agent *society*. Hence, whereas artificial intelligence studies intelligent agents, multi-agent systems studies *systems* of intelligent agents.

Examples of multi-agent systems and agents are plentiful. For instance, [57] mention the following examples of agents: "negotiators in a bargaining situation, communicating robots, or even components such as wires or message buffers in a complicated computer system". Furthermore, due to the advancement of the internet, most modern software is required to deal with autonomous interacting processes, which makes the multi-agent paradigm immensely appealing.

In addition to the fact that agents are situated in an environment of other agents, they usually also only have limited information. In other words, agents are typically not omniscient; they have local views of the world and thus only possess partial knowledge of the world. In this way they resemble real human agents. The resemblance with human agents also reveals the usefulness of knowledge and information: we know from our everyday life that information, knowledge, and beliefs play an important role in corporation, coordination, and negotiation. The same is the case in multi-agent systems. [57] provides additional examples of how reasoning about knowledge plays a role in multi-agent systems. Thus, it may come as no surprise that logics that deal with knowledge and beliefs are useful in multi-agent systems.

Because of the social aspect of multi-agent systems, communication among agents plays an important role. However, for communication to be possible between artificial agents, they need to speak a common language. By a common language, more is meant than just agents sharing a syntax, they also need to share a semantics – they need to be talking about the same things. This is one motivation for the development of common semantic frameworks, such as *formal ontologies*, which specifies a set of terms used to talk about a domain [172]. An example is the medical ontology SNOMED CT (Systematized Nomenclature of Medicine-Clinical Terms), which is a collection of over 311,000 structured medical terms with meaning [102]. Moreover, ontologies, and ontology languages such as OWL, are heavily involved in the idea of the *semantic web*, namely that information on the internet should be equipped with "meaning", see [100]. In addition to being a necessity for communication, the search for common semantic frameworks is the concern of "knowledge representation and reasoning", usually regarded as a subfield of artificial intelligence. The ability to represent vast amounts of knowledge in a way that is understandable by many and easy to access, is an advantage that goes well beyond applications in multi-agent systems. A standard example is the development of electronic patient files within healthcare. Due to the enormous quantity of concepts in the medical world and the fact that doctors, nurses, pharmacists, etc. may all use the concepts differently, it is an example with great complexity as well as large potential benefits.

The task of forming common semantic frameworks to make communication possible is something currently not discussed in social epistemology. However, it is a prerequisite for the exchange of information in a society, and therefore it deserves more attention within the social epistemology community. Nevertheless, philosophers are showing increased interest in constructing formal ontologies, especially in connection with biomedical sciences as exemplified by the collection [121]. For more on the relation between ontology in philosophy and computer science see [142]. How logic is used for knowledge representation through ontologies is further discussed in Section 1.3.2.1.

Another common problem in multi-agent systems is the need for agents to aggregate their preferences, judgments, beliefs, etc. [3]. For instance if a set of agents needs to coordinate an action, they need somehow to agree on taking that action. One such way of coming to agreement is through voting. Another example is an agent receiving different information from different sources who has to make up his mind on what to believe. Finally, the problem of judgment aggregation is yet another example where agents need to aggregate their judgment on some propositions.

A final note on the relationship between social epistemology and multi-agent systems is in order. Viewing social epistemology as being only about human agents and viewing multi-agent systems as being only about artificial agents makes the two fields disjoint. However, these are narrow views of the fields and in reality the two fields deal with many of the same topics and even use the same methods cf. the problem of judgment aggregation. Overall, information, knowledge, and beliefs are concepts that play a key role in both disciplines and moreover, they share a logic toolbox to model these concepts.

As previously mentioned, the need for tools that can help us find, understand, manage, and communicate information automatically on larger scales is great. Logic is a tool that through multi-agent systems can help in this process as, for instance, it allows for a better structuring of information through knowledge representation, for retrieval of new information through automated theorem proving, and for verification of information handling systems. Furthermore, logic may help clarify concepts related to information, knowledge, and beliefs, as they are discussed in epistemology. A more detailed discussion of the usefulness of logic in modeling information, knowledge, and beliefs is the topic of the next section.

1.3.2 Logic-based modeling of information, knowledge, and beliefs

Information plays a role in almost every science. However, when it comes to actual theories of information, Shannon's Mathematical Theory of Communication, dating back to [138], seems to be the first. Since then many different theories of information have emerged and existing theories have incorporated information as an important concept, most of them are described or mentioned in [2]. There is currently no unified theory of information, even though the different theories have much in common. Information seems such a broad concept that no unified theory is lurking around the corner. Using the terminology of [1], the different theories can be divided into large-scale views of information, such as in physics, kolmogorov complexity, or Shannon information theory, and small-scale views of information, regarding questions such as how individual agents possess, handle, and exchange information. Logic as a theory of information mostly regards the small-scale view [1].

Information shows itself in many forms in logic. The most direct form is the one it takes in epistemic logic. In epistemic logic agents explicitly reason about knowledge (and beliefs) and since knowledge involves information, epistemic logic certainly deals with information. The way epistemic logic deals with information is by viewing it as "range", to use the word of Johan van Benthem [1, 155].[46] As mentioned in Section 1.1.4, the "information as range" view is exemplified in epistemic logic by the definition of knowledge: φ is known to an agent if φ is true in all the worlds the agent considers possible. If an agent receives new information and thereby gains new knowledge, then the set of worlds he considers possible is decreased – i.e. information gain equals a shrinking of the "range". Thus, knowledge is defined explicitly in epistemic logic, and the same goes for beliefs (whether or not one uses plausibility models to define beliefs). Epistemic logic has another advantage when it comes to modeling knowledge, which is the fact that higher-order knowledge is easily dealt with. How epistemic logic allows for reasoning about higher-order knowledge and why it is useful in modeling knowledge, beliefs or information will be returned to in Section 1.3.2.3.

Another place where information shows up within logic is in connection with the syntactic or proof theoretic aspects of logic. A proof system for a logic takes syntactic input in the form of formulas and outputs new formulas that follow from the input without involving the semantics of the formulas. This can be viewed as a way of extracting some of the information contained in the premises [155]. Alternatively it can be viewed as a way of obtaining new information from already given information (expressed by the premises). These are two different views on the informativeness of deduction. The question of whether deductions produce new information or not is a deep philosophical question also known as *the paradox of deduction* or *the scandal of deduction* [66]. In one sense deduction seems to convey no new information that is not already contained in the premise, since the conclusion of a deduction follows with logical necessity from the premises. However real-life cases, where deduction provides new information, are plentiful. When it comes to computationally bounded agents such as humans or computers, deduction can indeed be useful and seems to be able to provide new information. We will not go into the philosophical discussion of the paradox of deduction, but merely notice the usefulness of deduction, at least for resource-bounded agents. The usefulness

[46] The "Information as range" view can be generalized beyond epistemic logic. In a given logic specified with respect to a semantic, the information contained in a formula can be viewed as the class of models that satisfies it, [155]. Thus, gaining information corresponds to decreasing the class of models, just as gaining information corresponds to eliminating possible worlds in epistemic logic.

of (automated) deduction will be further discussed in Section 1.3.2.2.

The reason why information is useful is because it is about something [66]. Focusing on the aboutness of information, [155] mentions yet another way where information shows up within logic, namely in situation theory. Here information is viewed as correlation. A situation can carry information about another situation because there is a correlation between the situations.[47] Keeping focus on the aboutness of information, there is another aspect of logic that deals with information which is not mentioned in [155]. This is the fact that logic is highly useful in structuring information – logical languages are exceptionally convenient for representing knowledge. A merely syntactic language cannot be used to represent information without the aboutness getting lost; the language needs to be accompanied by semantics. However, logic deals with developing formal languages with associated semantics and is thereby ideal for representing knowledge. Furthermore, as previously discussed, a common language is a prerequisite for successful communication or transfer of information. This requires an agreement of the used terms and once again the formal languages of logic are useful. The usefulness of logic in structuring information will now be elaborated.

1.3.2.1 Structuring information in formal ontologies

The importance of representing domain knowledge in a formal language, both for storing information as well as for making communication possible, should be clear by now. From the section introducing description logic (i.e. Section 1.1.3) it might also be obvious how a logic like description logic is useful in representing domain knowledge. Still, it is appropriate in the light of Chapter 7 to go into more detail on these matters.

Representing knowledge can be done, and has been done, in many ways within computer science as witnessed by the large field “knowledge representation and reasoning”. One way to represent knowledge is through ontologies, and especially formal ontologies have received great attention within computer science. There is no agreed, concise definition of what an ontology is, however, the following comes close; an *ontology* is a specification of a collection of terms and relations between them used to describe a particular domain, or with the words of [77]: “An *ontology* is an explicit specification of a conceptualization”. For ontologies to be implemented in computer systems they need to be made

[47] Viewing information as correlation and as range can be combined in a epistemic logic about situations, see [155].

formal, i.e. they need to be specified in a formal language, which again can be done in several ways. One widespread formal language, especially developed for the semantic web, is OWL [100]. Another way to specify an ontology is to use a formal logic such as description logic.[48] From the viewpoint of description logic, an ontology is simply a TBox.[49]

In line with the semantics of description logic, the world can be viewed as consisting of *individuals* (or instances), *concepts* (or classes) of individuals, and *relations* between individuals. An individual like a person Mary can be an instance of the concept Woman and stand in a special relation $\mathsf{has\_mother}$ to her mother Ann, expressed in description logic as $\mathsf{has\_mother}(\mathsf{Mary}, \mathsf{Ann})$. Now, the class Woman is a subclass of the class of persons, expressed in description logic as $\mathsf{Woman} \sqsubseteq \mathsf{Person}$. Furthermore, the class Person has the special property that every one of its members stands in the $\mathsf{has\_mother}$ relation to at least one person (actually exactly one) who is a woman, which again in description logic can be expressed as $\mathsf{Person} \sqsubseteq \exists \mathsf{has\_mother}.\mathsf{Woman}$. Thus, the formula $\mathsf{Person} \sqsubseteq \exists \mathsf{has\_mother}.\mathsf{Woman}$ expresses a relationship between the two classes Person and Woman, and so does $\mathsf{Woman} \sqsubseteq \mathsf{Person}$. Such relationships between classes are also referred to as *class relations*.

Ontologies are becoming increasingly important for the biomedical sciences since the vast amount of information they deal with can be systematized in large formal ontologies. Again the medical ontology SNOMED CT is an illuminating example, which as previously mentioned has more than 311,000 unique concepts and approximately 1,360,000 relationships between concepts [102]. For instance SNOMED contains the concept $\mathsf{Hepatitis}$, which in description logic syntax is defined by ([136]):

$$\mathsf{Hepatitis} \equiv \mathsf{Inflammatory\_disease} \sqcap \exists \mathsf{has\_location}.\mathsf{Liver}$$

With the enormous size of SNOMED CT the possibility of contradictions arising every time changes are made to the ontology is quite likely. For this reason, before releasing a new version of SNOMED CT, a description logic reasoner is used to check that the ontology remains consistent. Furthermore, in local applications of SNOMED CT new concepts may need to be added.

[48] The standard syntaxes of description logic and OWL are different, however, one version of OWL, namely OWL DL, has been made to correspond exactly to description logic. Thus, the difference between using OWL and using description logic is not so important.

[49] Since there is no agreed definition of what an ontology is, it is possible to view an ontology as also consisting of an ABox, however, viewing an ontology as only being a TBox seems to be the common view, where the union of a TBox and an ABox are usually referred to as a knowledge base.

When defining a new concept into the ontology it is possible that the new concept will have derived properties not explicitly given by the definition. Due to the size of the ontology, the only way to find out if this is the case is through automated reasoning using a description logic reasoner for instance. The tasks of checking whether SNOMED CT is consistent, finding derived properties of new concepts, as well as computing the entire subsumption hierarchy of SNOMED CT are examples of the description logic reasoning tasks described in Section 1.1.3. In addition, description logic can be used to infer new relationships between classes that may not be explicitly known to anyone yet.[50]

There are apparently great advantages in using logic to deal with large ontologies. However, before logic can be put to work, a formal ontology representing the domain needs to be constructed. This, in itself, can be a difficult task. One job that can obviously take a great deal of work is to collect all the information that needs to go into the ontology. Nevertheless, before this can be done at all, the domain in question needs to be clarified to such an extent that a formal language like description logic or OWL can be used to represent knowledge of the domain. Clarifying the domain and tailoring in detail a formal language to be able to describe interesting knowledge about a domain, is something that can require intensive philosophical consideration. However, the more expressive and precise our language is in talking about some domain, the easier it is to represent, obtain, and communicate useful information about the domain. Ideally, electronic patient journals will contain useful information in the sense that not only doctors, nurses, and other medical staff at the hospital can all understand it, but also doctors, nurses etc, at other hospitals and in other countries, so that they too can deliver the right treatment for the patient.

In Chapter 7 such philosophical considerations are applied to the domain of regulatory relations, especially as they appear in biomedical pathways. Regulatory relations are viewed as relations between classes such as in the statement “insulin positively regulates glucose transport”. Usually when representing relations between a class C and a class D, one of the following two formulas of description logic is used:

$$C \sqsubseteq \exists R.D \quad \text{or} \quad C \sqsubseteq \forall R.D.$$

[50]Description logic is extremely useful when dealing with a large ontology such as SNOMED CT, however, it is possible to use other logics to reason about an ontology such as SNOMED CT. Still, description logic remains one of the dominating logics to reason about formal ontologies.

However, based on philosophical considerations and specific domain knowledge, it is argued in Chapter 7 that none of these adequately represent regulatory class relations. Instead first-order logic is used and the chapter discusses how to extend classical description logics to capture regulatory class relations as well.

1.3.2.2 Automated reasoning

Logic aims at creating formal models of reasoning to distinguish "correct" reasoning from "wrong" reasoning. Proof theory then provides methods for determining whether an argument constitutes correct reasoning, as discussed in Section 1.2. Furthermore, in numerous cases, the determination of correct reasoning can be *automated* to such an extent that computers can be programmed to do the job. (For undecidable logics such programs are not guaranteed to give an answer to every input, of course.) In general, automated reasoning is the study of computer-assisted reasoning.

The significance of automated reasoning when dealing with ontologies and knowledge representation was discussed in detail in the last section; however, automated logic reasoning is useful in various other areas. In this section, other uses of automated reasoning are briefly discussed and connections are made with the proof theoretic issues presented in Section 1.2.

Disciplines where automated reasoning is widely used include specification and verification of hardware and software systems. Formal verification of hardware systems is used in industry, for instance to ensure that microprocessors work correctly, whereas formal verification of security protocols is an example of software verification. Hardware as well as software systems play essential roles in numerous activities, however, the complexity of such systems increases the likelihood of errors occurring in the systems. To avoid such errors, systems can be formally verified to meet their design specifications and thereby exclude the possibility of errors. A *specification* is a description of some desirable properties of a system, whereas a formal *verification* (or a proof of correctness) of a system is a formal mathematical proof that the system actually satisfies the required specifications. However, for such a formal proof to be possible, the specifications need to be properly presented in the same formal language as the proof is to be carried out in. Correctness of programs and hardware can only be verified with respect to a given specification – to answer the question of whether a program behaves correctly we need to have specified formally what it means for a program to "behave correctly". Note that a specification does not specify how a system should be designed to obtain its desirable properties – this is a completely different story.

Logic provides several formal languages to represent specifications as well as a rich proof theory to find such formal proofs. Nevertheless, the "proofs" of proof theory are not the only way to provide formal proofs of correctness of systems, model checking can be used as well – a system can be abstracted to a model of a particular logical language in which the specifications are stated, after which it can be checked whether the model satisfies the specifications. Either way, logic has shown itself useful in specifying and verifying software and hardware systems which include multi-agent systems [172].

As mentioned in Section 1.1.5 dynamic epistemic logic has been used for verification of security protocols for communications in multi-agent systems [164, 50, 51, 95, 4]. For instance, [164] use dynamic epistemic logic for automatic verification of anonymity in security protocols. Specification of protocols in multi-agent systems was also discussed in the standard reference [57]. [51] provides an overview of the use of epistemic logic in the analysis of security protocols. See [172, 166] for other examples of the use of automated reasoning and logic in specification and verification of multi-agent systems.

It is beoynd the scope of this thesis to design automated reasoning systems for the introduced logics, however, the work of chapters 2 and 5 are first steps towards automatic reasoning. Chapters 2 and 5 provide terminating tableau systems for three logics, and at the end of Section 1.2.1 it was sketched how such terminating tableau system can give rise to decision procedures. Even though such decision procedures are an important steps towards automated reasoning for the given logics, there is the non-trivial task of implementing the decision procedures. Implementing the decision procedures resulting from the tableau systems of chapters 2 and 5, is beyond the scope of this thesis. Nevertheless it should be mentioned that the tableau systems of [37] discussed in Section 1.2.2 have been implemented in for example the HTab theorem prover [98]. Hence, implementation of the tableau systems of chapters 2 and 5 might be done as extensions of HTab.

An implemented decision procedure is not a final objective for automated reasoning in itself. If a decision procedure is implemented, an answer to the question of validity of a formula is guaranteed, however, the answer might not be provided in a short time, it might even take longer than the time the universe is expected to exist. This leads to the issue of tractability, and the task of optimizing implementation to make it more tractable. However, there might be a theoretical bound on how much optimizations can be achieved, coming from the *computational complexity* of a logic. Such issues are important and intensely studied, but beyond the scope of this thesis as well.

As observed, there are several issues to consider when developing automated reasoning, and other proof systems than tableau systems can be the starting point as well – tableau systems are not the only way or always the best way necessarily. Furthermore, model checking is another important task for automated reasoning. In general, automated reasoning is a large field that tackles several important issues not discussed here and uses a variety of other advanced methods. Still, automated reasoning has many applications in computer science beyond the already mentioned applications in formal ontologies and knowledge representation as well as the applications in specification and verification of hardware and software systems. In addition, automated reasoning has applications in philosophy and mathematics as well. See [129] for a general introduction to automated reasoning as well as further applications.

1.3.2.3 Reasoning about knowledge about knowledge about...

Knowledge or information is important for agents when deciding which actions to adopt. However, knowledge about other agents' knowledge can also be important in a multi-agent setting when agents have to cooperate, coordinate, and negotiate. In this section the importance of reasoning about other agents' information, knowledge, and beliefs is elaborated further.

Computer science contains numerous examples of the usefulness of reasoning about knowledge. For instance, in security protocols for communications in multi-agent systems, a receiver reasons about whether a sender knows that he knows the content of the message. Another related example is the *Russian card problem* [161, 163], which shows the importance of correct reasoning about other agents' knowledge in designing cryptographic protocols. In the Russian card problem seven cards (denoted by $0, 1, 2, 3, 4, 5, 6$) are distributed between three agents a, b, and c, such that a receives 3 cards, b receives 3 cards, and c receives the final card. The problem is now to design a protocol specifying how agent a and b can inform each other about their own cards through truthful public announcements (also perceivable by c) in such a way that afterwards a knows all of b's 3 cards, b knows all of a's 3 cards, and c knows none of a's or b's cards.[51] Assume that actually a has cards $0, 1, 2$ and b has cards $3, 4, 5$. If a announces that "I have $0, 1, 2$ or b has $0, 1, 2$" and b

[51] It is assumed common knowledge that all agents know that a has 3 cards, b has 3 cards, c has 1 card, and all the cards are among the cards $0, 1, 2, 3, 4, 5, 6$.

afterwards announces that "I have $3, 4, 5$ or a has $3, 4, 5$", it might seem that the problem has been solved, however, the first announcement can only be made safely by a if she has $0, 1, 2$, which c can use to learn that a has $0, 1, 2$, [163]. The non-trivial insight that this does not constitute a solution can be made clear with a formalization of the problem and the solution into dynamic epistemic logic, see [161, 163]. Thus, dynamic epistemic logic is a powerful formalism to represent reasoning about knowledge.

Another classic example is the *Muddy children puzzle* (see for instance [57, 163]): A group of n children have been playing outside in the mud and $k \geq 1$ of them have become muddy on their forehead. The children can see the mud on the other children's forehead, but they cannot see whether they have muddy foreheads themselves. The father then announces to them that at least one of them is muddy. He then asks all of those who know they are muddy to step forward. If no one steps forward, he repeats his request. As long as none of the children step forward he keeps repeating the request in this way. If the children are all perfect logical reasoners all the muddy children will step forward after the father has repeated the request k times. Again, this can be modeled in epistemic or dynamic epistemic logic, see [57, 163]. In this example it is essential that the children reason about the other children's knowledge, the other children's knowledge about their knowledge, the other children's knowledge about their knowledge about the other children, and so on. Hence, reasoning about several "levels" of knowledge can be quite essential.

Yet another classical example of why reasoning about knowledge is important is a traffic situation: It is not enough that a driver knows what the red and green lights mean, he must also know that all other drivers know this in order for him to safely decide to drive on green. However, that everyone knows that everyone knows what red and green means is not enough. For the convention of driving on green and stopping on red to really kick in, it has to be common knowledge what the red and green lights mean. This was claimed by David Lewis who was the first to introduce a notion of common knowledge in the philosophical literature in his famous book on conventions [109], where he defined conventions as a particular kind of solution to co-ordination games [170]. Since then, the importance of common knowledge in various coordination problems has been widely recognized. For further examples see [167].

In the economic literature common knowledge was independently invented by Robert Aumann in his famous paper [10].[52] Here Aumann showed that if

[52]The notion of common knowledge introduced by Lewis is not a formal notion and it is

two agents have the same prior distribution (i.e. agree on how the world looks) and an event takes place (which might give the agents different information), then; if their posterior distributions are common knowledge among them, then they must have the same posterior distribution (if it is common knowledge how they each see the world after the event they must see the world in the same way) – they cannot agree to disagree. Both Lewis and Aumann saw the great importance of common knowledge, which is a concept that involves reasoning about higher-order knowledge to a great extent, since common knowledge that φ amounts to everybody knowing φ, everybody knowing that everybody knows φ, ... and so on – according to the standard definition.

In game theory, not just common knowledge, but knowledge in general has played an important role in justifying game equilibriums. An equilibrium (or a solution) of a game represents a systematic description of a possible outcome of the game, an example is the well-known *Nash* equilibrium. However, a solution of a game does not specify what the agents will actually do [124]. Viewing game theory as being about deliberate strategic interactions of rational agents, an explanation is needed as to why the mathematically defined notions of equilibriums should ever occur. Typically, such an explanation is sought in the players' rationality and their knowledge about the game and the other players. "Epistemic game theory" (also referred to as "interactive epistemology") investigates which kind of knowledge conditions need to be satisfied for the game to end up in an equilibrium. For instance, common knowledge of rationality may lead to the so-called backward induction solution of a game, but weaker conditions may also lead to equilibriums. Again reasoning about knowledge and other agents' (or players') knowledge are important and it naturally leads to the use of epistemic logic in game theory. For an overview of the use of modal and epistemic logic in game theory, see [160].

When formalizing solution concepts based on knowledge in games or disagreement theorems it has been realized that the usual epistemic or dynamic epistemic logics are not expressive enough – machinery from hybrid logic is needed, [158, 133, 134, 47]. This provides one motivation for the enterprise of combining dynamic epistemic logic and hybrid logic, as in chapters 4 and 5.

Even though game theory deals with models of how agents interact and how information, knowledge, and beliefs matter for interaction, there is still a large gab between the idealized models of game theory and actual human

a bit different from the notion of common knowledge formally defined in Section 1.1.4, see for instance [167]. The notion of common knowledge presented in Section 1.1.4 agrees with Aumann's definition in [10].

agents. The same is the case for traditional epistemic logic. However, there is a trend that the idealized models of game theory and epistemic logic are returning to issues of real human behavior inspired by for instance psychology [151, 168].

In addition to this, social epistemology also contains examples of the formal methods of logic being put to use in modeling real social phenomena. How information is exchanged among agents in a social context and how agents update their knowledge and beliefs based on communication is a hot topic in epistemic logic for intelligent interactions. The way such agents update their beliefs under new incoming information depends on how well they trust the source of information. This is a clear example of a meeting between epistemic logics and social epistemology, since social epistemology is greatly interested in the role of trust and testimony in knowledge acquisitions. An example of using dynamic epistemic logic to model updates of an agent's beliefs based on his trust in the information source, is [99]. Yet another example from social epistemology of modeling the exchange of information among agents in a social context is the study of pluralistic ignorance [46, 91]. It is in this context that Chapter 6 should be viewed.

In Chapter 6, the logic of plausibility models introduced in Section 1.1.4 is used to investigate different versions of pluralistic ignorance as to whether they are logically consistent and fragile phenomena. (A phenomenon is considered fragile if it easily dissolves due to certain changes.) Pluralistic ignorance can be viewed as a social norm, for instance the social norm of drinking among students on American college campuses, and social norms can be defined as game equilibriums in similar manners as conventions were defined by Lewis, see for instance [21].[53] However, in Chapter 6 pluralistic ignorance is viewed purely as an informational phenomenon, which makes it an object for epistemic logic modeling. The chapter only contains a first approach to a logic-based model of pluralistic ignorance, nevertheless the phenomenon seems to involve many interesting aspects of how information flows between agents in a social context. For instance, maybe if the way agents acquire higher-order information is included in the model, pluralistic ignorance might turn out to be a rational social phenomenon, and not another example of human irrationality and illogicality. However, such speculations are still left for future research.

[53]Note that social norms and conventions need not be the same thing.

1.3.3 Judgment aggregation and many-valued logics

In decision-making the right information is vital, but sometimes agents are not alone in making the decisions, a joint action might be required of a group of agents. In such a case the agents can pull their information together to gain new information that might help them perform the best group action. However, in many cases of decision-making, agents only have partial information or they might have contradicting information as a group. Thus, the agents can have different beliefs about the information on which they base their decisions, but for a joint action to be possible they might be required to agree on a joint set of information. A classic example is a jury deciding on a verdict. These issues lead to the study of judgment aggregation.

When introducing the many-valued logic of chapters 2 and 3 back in Section 1.1.6, the original motivation for the logic given by Fitting was briefly mentioned, namely the motivation that the truth value of a formula can be identified with the set of agents that accept that formula. In this section, this motivation is described in more detail before the idea is put to work on the problem of judgment aggregation and more generally the problem of aggregating propositional attitudes.

1.3.3.1 Sets of agents as truth values

The simple idea of Fitting to "take the truth value of a formula to be the set of agents that accept the formula as true" [63], dates back to his papers [59, 60, 61] of the early 1990s. Given a finite set of agents $\mathbb{A}$, the power set $\mathcal{P}(\mathbb{A})$ can be used as a truth-value set. Thus, any formula φ is assigned a truth value from $\mathcal{P}(\mathbb{A})$ corresponding to the set of agents that accept φ as true. It has to be noted that such an assignment of truth values actually is "consistent", in the sense that if φ is assigned the set $\{a, b, c\}$ and ψ the set $\{a, c, d\}$, then $\varphi \wedge \psi$ is assigned $\{a, c\}$, $\varphi \vee \psi$ is assigned $\{a, b, c, d\}$, and so on. However, this can easily be assured, by only assigning truth values to the propositional variables based on the agents that accept them, and then using the operations of intersection $\cap$, union $\cup$, and complement $\overline{\cdot}$ on $\mathcal{P}(\mathbb{A})$ to interpret complex formulas built up from $\wedge$, $\vee$, $\neg$, and $\rightarrow$.[54] In this process, it is revealed that the Boolean algebra structure of $\mathcal{P}(\mathbb{A})$ ensures that this can be done in a natural way. Thus, this leads to investigation of many-valued

[54] In interpreting the formula $\varphi \rightarrow \psi$ the fact that it is equivalent to $\neg\varphi \vee \psi$ is used.

logics where the set of truth values are finite Boolean algebras.[55]

In the more general setting of Heyting algebras, Fitting realized that this idea can be extended to modal logic as well. Using the operations on a Boolean algebra it is possible to define the semantics of $\Box$ and $\Diamond$ in such a way that the truth values of the formulas $\Box\varphi$ and $\Diamond\varphi$ exactly become the sets of agents that accept them as true. Adopted from [63] the truth values $\nu(w,\cdot)$ at a world w in a model $\mathcal{M} = \langle W, R, \nu\rangle$[56], are:

$$\nu(w, \Box\varphi) = \bigcap_{v\in W} \Big(\overline{R(w,v)} \cup \nu(v,\varphi)\Big),$$

$$\nu(w, \Diamond\varphi) = \bigcup_{v\in W} \Big(R(w,v) \cap \nu(v,\varphi)\Big).$$

That this definition ensures that the truth values of $\Box\varphi$ and $\Diamond\varphi$ become exactly the set of agents that accept the formulas requires a few precise definitions and a mathematical proof. These are left out here, but can be found in [63]. That the truth values of $\Box\varphi$ and $\Diamond\varphi$ are exactly the set of agents that accept the formulas is the main motivation for the given semantics of the modalities.

Using the notion of relative pseudo-complements in finite Heyting algebras, the many-valued semantics based on Boolean algebras can be generalized further to Heyting algebras as introduced in Section 1.1.6 – this was the original idea of Fitting. Adopting the Heyting algebra notation introduced in Section 1.1.6, the semantics of $\Box\varphi$ and $\Diamond\varphi$ then become:

$$\nu(w, \Box\varphi) = \bigsqcap_{v\in W} \Big(R(w,v) \Rightarrow \nu(v,\varphi)\Big),$$

$$\nu(w, \Diamond\varphi) = \bigsqcup_{v\in W} \Big(R(w,v) \sqcap \nu(v,\varphi)\Big),$$

where $x \Rightarrow y$ is the relative pseudo-complement of x with respect to y, $\bigsqcap$ is the meet, and $\bigsqcup$ is the join as introducsed in Section 1.1.6. Since a Heyting algrebra does not have a complement operation, $\overline{R(w,v)} \cup \nu(v,\varphi)$ needs to be replaced by $R(w,v) \Rightarrow \nu(v,\varphi)$, however the Heyting algebra is a Boolean algebra as well the two definitions coincide.

[55]Every powerset structure $(\mathcal{P}(A), \cup, \overline{\cdot}, \emptyset)$ is a Boolean algebra. Thus, logics with sets $\mathcal{P}(\mathbb{A})$ as truth values can be investigated by studying logics with Boolean algebras as truth values. The link between powerset structures and Boolean algebras is even stronger since Stone's representation theorem shows that every Boolean algebra is isomorphic to a subalgebra of a power set algebra of the form $(\mathcal{P}(A), \cup, \overline{\cdot}, \emptyset)$ [27].

[56]Here W is just a non-empty set, $R : W \times W \to \mathcal{P}(\mathbb{A})$, and $\nu : W \times \mathsf{PROP} \to \mathcal{P}(\mathbb{A})$.

Taking the set $\mathcal{P}(\mathbb{A})$ as the set of truth values naturally leads to a Boolean algebra, but how do Heyting algebras naturally arise when keeping the underlying idea that the truth value of a formula corresponds to the set of agents that accept the formula? Heyting algebras naturally arise if there is a relation of dominance among the agents. An agent a is said to dominate an agent b if whenever a accepts a formula φ, b also accepts φ. Note that this definition has an intuitionistic flavor[57] – otherwise the set of formulas accepted by b would be identical to the set of formulas accepted by a. Accordingly, agent b can accept a formula without a accepting it, but if a accepts it so does b. Therefore, the set of truth values based on the two agents a and b will not be the entire Boolean algebra $\mathcal{P}(\{a, b\})$, but instead the Heyting algebra consisting of the three elements $\{\emptyset, \{b\}, \{a, b\}\}$ – if a dominates b, the truth value $\{a\}$ is never assigned to any formula. For the technical details and a proof that the notion of dominance between agents leads to Heyting algebras as truth values see [60].

1.3.3.2 Judgment aggregation

The best way to understand the subject of judgment aggregation is through its main research question as formulated by Christian List [110]: "How can a group of individuals make consistent collective judgments on a set of propositions on the basis of the group members' individual judgments on them?" A classical example of a judgment aggregation problem is a jury who has to give a verdict on a case viewed as a set of propositions. The rest of this short introduction to judgment aggregation will mainly be based on [110].

The motivation for studying judgment aggregation mainly comes from paradoxes involving majority voting. Such paradoxes show that majority voting can generate collective judgments that are inconsistent, and they are often referred to as *discursive dilemmas*. A standard example of a discursive dilemma form the judgment aggregation literature is rather illustrative for the issues involved. Assume there are three agents a, b, c that have to make judgments on the three propositions $p, q, p \to q$ (i.e. they have to specify whether they judge each of them to be true or false). Assume that the agents make

[57] There is a close relation between the many valued logic and intuitionistic logic, see [60] and Section 2.2.3 of Chapter 2.

the following judgments:

	p	q	$p \to q$
Agent a	true	true	true
Agent b	true	false	false
Agent c	false	false	true
Majority voting	true	false	true

Here a and c judge $p \to q$ to be true and thus a majority vote on this proposition would yield a collective judgment of $p \to q$ being true. However, if majority voting is used to obtain collective judgments on p and q, p is judged to be true and q is judged to be false, and $p \to q$ thereby becomes false. Hence, majority voting on each proposition can lead to an inconsistent collective judgment.

After this illustrative example it is time to make the problem of judgment aggregation mathematically precise (using the notions from [110]). Assume a finite set of agents $\mathbb{A} = \{a_1, ..., a_n\}$ to be given, and assume a logic that extends classical propositional logic to be given with an associated language $\mathcal{L}$ and a notion of consistency. For any formula $\varphi \in \mathcal{L}$, let $\sim\varphi$ be $\neg\varphi$ if φ is not of the form $\neg\psi$, and if φ is of the form $\neg\psi$ then let $\sim\varphi$ be ψ. An *agenda* is any subset $X \subseteq \mathcal{L}$ satisfying; $\varphi \in X$ iff $\sim\varphi \in X$. In the following X will be assumed to be some fixed agenda. The intuition is now that every agent in $\mathbb{A}$ makes a judgment on the propositions in the agenda X, and the problem of judgment aggregation is then to give a procedure (or rule) for aggregating their individual judgments into one collective judgment on the propositions in X. The statement "agent a makes a judgment on $\varphi \in X$", is usually taken to mean that a believes φ to be true or a believes φ to be false. For an agent a, the judgment of a on the propositions in X will be represented as a subset $J_a \subseteq X$, corresponding to the set of formulas in X that a judges to be true. A tuple $(J_{a_1}, ..., J_{a_n})$ (remember that $\mathbb{A} = \{a_1, ..., a_n\}$) will be referred to as a *profile*. In general, any subset $J \subseteq X$ is called a *judgment set*, and the set of all judgment sets will be denoted by $\mathbf{J}$. Then, $\mathbf{J}^n$ will be the set of all possible profiles. A judgment set is said to be *consistent* if it is a consistent set according to the given logic, and it is said to be *complete* if for any $\varphi \in X$ it contains either φ or $\sim\varphi$. Finally, an *aggregation rule* is a function $F : \mathbf{J}^n \to \mathbf{J}$. The problem of judgment aggregation can now be formally stated as the problem of finding an aggregation rule F such that for any profile $(J_{a_1}, ..., J_{a_n})$, $F(J_{a_1}, ..., J_{a_n})$ is a consistent judgment set.

Obviously, it is natural to put some further requirements on the input, the output, and the responsiveness of a judgment rule that is a candidate for solving the problem of judgment aggregation. For instance, a natural requirement on the input is that the agents' judgment sets are consistent and complete, i.e. that the domain of F is precisely the set of all consistent and complete judgment sets denoted by **CCJ**. This condition is known as *universal domain.* Another natural and essential requirement is of course that the output is also a consistent and complete judgment set, i.e. that the range of F is **CCJ**. This condition is known as *collective rationality.* Other natural requirements on the responsiveness of a judgment rule are the so-called *systematicity* and *anonymity* requirements (see [110, 111]). Now, [111] showed that under some mild conditions no judgment rule exists that satisfies universal domain, collective rationality, anonymity, and systematicity simultaneously. Another classical impossibility result from [127] states that under some other natural requirements the only possible judgment rules are dictatorships, where a judgment rule F is a *dictatorship* if there is some agent a_i such that $F(J_{a_1}, ..., J_{a_n}) = J_{a_i}$ for all profiles $(J_{a_1}, ..., J_{a_n})$. Such impossibility results have dominated the literature on judgment aggregation. In spite of these mostly negative results in judgment aggregation, there may still be room for some positive results. Viewing judgment aggregation through the eyes of many-valued logic might be a first step towards such a positive result, or maybe just another insight into the nature of the negative results.

1.3.3.3 A many-valued approach to judgment aggregation

Adopting the idea of Fitting, to take sets of agents as truth values, there seems to be an obvious way of aggregating agents' judgments into a collective judgment in a many-valued extension of the logic. Simply take the rule that assigns to a formula φ the truth value that corresponds to the set of agents that judge φ to be true. This idea will be elaborated on in this section as well as the relation to other many-valued approaches to judgment aggregation. For simplicity, the logic in this section will be constrained to propositional logic (the propositional language will be denoted by $\mathcal{L}_p$). In the next section some remarks on general modal and hybrid logics will be given.

Assume a set of agents $\mathbb{A} = \{a_1, ..., a_n\}$ to be given. The truth value set $\mathcal{P}(\mathbb{A})$ can then be formed. Define a *many-valued assignment* ν to be a function

$\nu : \mathcal{L}_p \to \mathcal{P}(\mathbb{A})$ that satisfies[58]:

$$\begin{array}{rcl} \nu(\neg\varphi) & = & \overline{\nu(\varphi)} \\ \nu(\varphi \wedge \psi) & = & \nu(\varphi) \cap \nu(\psi) \\ \nu(\varphi \vee \psi) & = & \nu(\varphi) \cup \nu(\psi) \\ \nu(\varphi \to \psi) & = & \overline{\nu(\varphi)} \cup \nu(\psi) \end{array} \tag{1.11}$$

Let the set of all many-valued assignments be denoted by V. Given an agenda X, a "canonical" many-valued judgment rule $F_c : CCJ^n \to \mathsf{V}$ can then be defined by requiring the many-valued assignment $F_c(J_{a_1}, ..., J_{a_n})$ to be given by:

$$F_c(J_{a_1}, ..., J_{a_n})(\varphi) = \{a \in \mathbb{A} \mid \varphi \in J_a\},$$

for every profile $(J_{a_1}, ..., J_{a_n})$. It is not hard to see that $F_c(J_{a_1}, ..., J_{a_n})$ will always satisfy (1.11) and thus be a well-defined many-valued assignment.

The definition of F_c is best understood through an example, for instance the example from the last section. There the agenda was $\{p, \neg p, q, \neg q, p \to q, \neg(p \to q)\}$, and $\mathbb{A} = \{a, b, c\}$, $J_a = \{p, q, p \to q\}$, $J_b = \{p, \neg q, \neg(p \to q)\}$, and $J_c = \{\neg p, \neg q, p \to q\}$. By the definition of F_c the aggregated truth values become:

$$\begin{array}{lcl} F_c(J_a, J_b, J_c)(p) & = & \{a, b\} \\ F_c(J_a, J_b, J_c)(\neg p) & = & \{c\} \\ F_c(J_a, J_b, J_c)(q) & = & \{a\} \\ F_c(J_a, J_b, J_c)(\neg q) & = & \{b, c\} \\ F_c(J_a, J_b, J_c)(p \to q) & = & \{a, c\} \\ F_c(J_a, J_b, J_c)(\neg(p \to q)) & = & \{b\}, \end{array}$$

where (J_a, J_b, J_c) is the profile given in the example of the last section. Note that $F_c(J_a, J_b, J_c)$ is a consistent many-valued assignment, in particular

$$F_c(J_a, J_b, J_c)(p \to q) = F_c(J_a, J_b, J_c)(\neg p) \cup F_c(J_a, J_b, J_c)(q).$$

In general, F_c always gives consistent and complete judgment sets as long as the domain is required to be the set $\mathbf{CCJ}^n$. This is not hard to see.

[127] discuss judgment aggregation in a many-valued logic setting as well. However, their sets of truth values are linear orders. As a consequence, in the example of the last section the truth values assigned to p, $\neg q$, and $p \to q$

[58]Note that a many-valued assignment that satisfies (1.11) is uniquely determined by its values on the propositional variables.

become the same, because they are all of the same "level" (the sets that constitute the truth values all have cardinality 2). However, if the truth value set is a Boolean algebra, the formulas p, $\neg q$, and $p \to q$ can all have a truth value at the same "level" without the truth values becoming identical. [52] present a general framework for aggregating propositional attitudes where many truth values are also allowed. In the first theorem of that paper the sets of truth values are linear orders as well, but in the second theorem the result is generalized to partial orders with minimal elements. However, it is hard to see how the logical connectives are to be interpreted in their framework. From a logical point of view it seems natural to put a minimal requirement of at least being a Heyting algebra, on the truth value sets.

In the many-valued approach to judgment aggregation presented here, there is no paradox since F_c always produces consistent and complete judgment sets. However, the judgment aggregation problem is not completely solved – it is merely transformed. Still, there might be some new insights from the many-valued approach presented. First, note that aggregating the individual judgments of a group of agents is usually done with the purpose that the group of agents needs to make a joint action. For instance, a jury needs to find the accused either guilty or not guilty – there are only these two possibilities. Still, there might potentially be cases where the choices of the group are more than just "two-valued" and thus the many-valued aggregation might be useful. For instance, a company considering whether to invest in new technology may not only consider whether to invest or not, but also consider how much to invest. This suggests a distinction between the set of truth values that governs the aggregation of judgments and the set of truth values that governs the subsequent decision-making problem. In addition, it suggests that it might be interesting to study judgment aggregation together with associated decision-making problems.

Consider the example of the last section once more. Assume that there is an associated decision-making problem that requires the agents to make a judgment of $p \to q$ being either true or false. Thus, the truth value set that governs the aggregation problem is the Boolean algebra $\mathcal{P}(\{a, b, c\})$, whereas the truth value set that governs the associated decision-making problem is the Boolean algebra $\{\mathit{true}, \mathit{false}\}$. The question is then whether the Boolean algebra $\mathcal{P}(\{a, b, c\})$ can be transformed into the Boolean algebra $\{\mathit{true}, \mathit{false}\}$ in a natural and satisfactory way such that the agents will assign either *true* or *false* to $p \to q$? In this example there is no obvious way of doing this, and it might even be impossible depending on how a "natural and satisfactory" transformation is defined. This is left for future research to decide.

In this setting, where the truth value sets are generalized to Boolean algebras, the problem of judgment aggregation is transformed into a problem of transforming Boolean algebras, more precisely it is turned into a problem of finding Boolean homomorphisms from one Boolean algebra to another. This way of viewing judgment aggregation is not entirely new, however. Frederik Herzberg has studied judgment aggregation from the viewpoint of Boolean algebra homomorphisms in several papers, for instance [54, 94]. A closer study of the combination of the many-valued approach to judgment aggregation and the work of Herzberg is left for future research as well.

1.3.3.4 A many-valued approach to aggregating propositional attitudes

In the many-valued treatment of judgment aggregation in the last section, only propositional logic with truth values in Boolean algebras were considered. The logic of Chapter 2 generalizes this logic in two ways. First of all the truth values come from a Heyting algebra and not a Boolean algebra, and secondly the logic is extended from merely propositional logic to a hybrid logic. The first generalization allows for the study of judgment aggregation in domains where there is a relation of dominance among the agents. The second generalization is interesting as well. Since hybrid logic is an extension of modal logic and modal logic is a logic for propositional attitudes, the many-valued hybrid logic may be useful in studying aggregation of propositional attitudes. The problem of aggregating propositional attitudes is also studied in [52].

The paper [52] discuss aggregation of propositional attitudes in a general framework. In the first part of the paper propositional attitudes are modeled by either propositions in a two-valued logic (judgments) or by probabilities. Given an agenda X, an agent a's propositional attitude is an attitude function A_a on X assigning values is some set V (for probabilities V is $[0,1]$ and for judgments V is $\{0,1\}$). Thus, that an agent a believes φ is modeled as $A_a(\varphi) = 1$, and that agent a believes φ to a degree $p \in [0,1]$ is modeled as $A_a(\varphi) = p$. In the second part of the paper the set V is a general partial order with a minimal element and the attitude functions are required to satisfy three minimal requirements. Still, there seems to be no obvious way to represent and aggregate higher-order propositional attitudes in framework of [52]. However, in the many-valued approach presented here, a generalization is made to all propositional attitudes that can be represented by a modal logic, and the logic thereby easily allows for higher-order propositional attitudes to be aggregated as well. Again, there is much future research laying ahead when it comes to modeling aggregation of propositional attitudes, and the many-valued hybrid logic of Chapter 2 might be an interesting way to go.

1.4 Outline of the thesis

This introduction should have given the reader a detailed idea about what the following chapters of the thesis contain. Therefore, there is no need for an extensive outline of the rest of the thesis, a simple list of the remaining six chapters should do:

- Chapter 2: "Many-Valued Hybrid Logic" joint work with Thomas Bolander and Torben Braüner, published as [83] and [84].
- Chapter 3: "Alternative semantics for a many-valued hybrid logic" unpublished manuscript.
- Chapter 4: "A Hybrid Public Announcement Logic with Distributed Knowledge" an extended version of a paper published as [81].
- Chapter 5: "Terminating tableaux for dynamic epistemic logics" published as [80].
- Chapter 6: "A Logic-Based Approach to Pluralistic Ignorance" published as [82].
- Chapter 7: "Logical Knowledge Representation of Regulatory Relations in Biomedical Pathways" joint work with Sine Zambach, published as [176].

Even though the formatting has changed, the content of the chapters are identical to the published papers, apart from minor corrections and clarifications. Therefore, the chapters may contain occasional overlaps. Finally, the thesis is ended by a short conclusion in Chapter 8.

Chapter 2

Many-Valued Hybrid Logic

(co-authored with Thomas Bolander and Torben Braüner)

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Abstract: In this paper we define a many-valued semantics for hybrid logic and we give a sound and complete tableau system which is proof-theoretically well-behaved, in particular, it gives rise to a decision procedure for the logic. This shows that many-valued hybrid logic is a natural enterprise and opens up the way for future applications.[1]

Keywords: Modal logic, hybrid logic, many-valued logic, tableau systems.

2.1 Introduction

Classical hybrid logic is obtained by adding to ordinary, classical modal logic further expressive power in the form of a second sort of propositional symbols called nominals, and moreover, by adding so-called satisfaction operators. A nominal is assumed to be true at exactly one world, so a nominal can be considered the name of a world. Thus, in hybrid logic a name is a particular sort of propositional symbol whereas in first-order logic it is an argument to a predicate. If i is a nominal and ϕ is an arbitrary formula, then a new formula $@_i\phi$ called a satisfaction statement can be formed. The part $@_i$ of $@_i\phi$ is called a satisfaction operator. The satisfaction statement $@_i\phi$ expresses that the formula ϕ is true at one particular world, namely the world at which the nominal i

[1] One such application could be judgement aggregation as discussed in Section 1.3.3.

is true. Hybrid logic is proof-theoretically well-behaved, which is documented in the forthcoming book [42]. Hybrid-logical proof-theory includes a long line of work on tableau systems for hybrid logic, see [25, 24, 39, 37, 79, 36].

Now, classical hybrid logic can be viewed as a combination of two logics, namely classical, two-valued logic (where the standard propositional connectives are interpreted in terms of the truth-values true and false) and hybrid modal logic (where modal operators, nominals, and satisfaction operators are interpreted in terms of a set of possible worlds equipped with an accessibility relation). The present paper concerns many-valued hybrid logic, that is, hybrid logic where the two-valued logic basis has been generalized to a many-valued logic basis. To be more precise, we shall define a many-valued semantics for hybrid logic, and we shall give a tableau system that is sound and complete with respect to the semantics. Not only is the many-valued semantics a generalization of the two-valued semantics, but if we chose a two-valued version of the many-valued tableau system, then modulo minor reformulations and the deletion of superfluous rules, the tableau system obtained is identical to an already known tableau systems for hybrid logic. Our many-valued semantics is a hybridized version of a many-valued semantics for modal logic given in the papers [59, 60, 61]. A notable feature of this semantics is that it allows the accessibility relation as well as formulas to take on many truth-values (in other many-valued modal logics it is only formulas that can take on many truth-values).

A leading idea behind our work is that we distinguish between the way of reasoning and what the reasoning is about, and in accordance with this idea, we generalize the way of reasoning from two-valued logic to many-valued logic such that we reason in a many-valued way about time, space, knowledge, states in a computer, or whatever the subject-matter is. Given our distinction between the way of reasoning and what the reasoning is about, we take it that the concerns of hybrid logic basically are orthogonal to as whether the logic basis is two-valued or many-valued. Thus, it is expectable that the already known proof-theoretically well-behaved tableau systems for two-valued hybrid logic can be generalized to proof-theoretically well-behaved tableau systems for many-valued hybrid logic. Accordingly, if we define a many-valued hybrid logic and give a tableau system that satisfies standard proof-theoretic requirements (it is cut-free, it satisfies a version of the subformula property, and it gives rise to a decision procedure), then we learn more about hybrid logic and we provide more evidence that hybrid logic and hybrid-logical proof-theory is a natural enterprise.

This paper is structured as follows. In the second section of the paper we define the many-valued semantics for hybrid logic and we make some remarks on the relation to intuitionistic hybrid logic. In the third section we introduce a tableau system, in the fourth section we prove termination, and in the fifth section we prove completeness.

2.2 A Many-Valued Hybrid Logic language

In this section a Many-Valued Hybrid Logic language (denoted by **MVHL**) is presented and a semantics for the language is given. We have included global modalities, one reason being that they are used in our motivation for our choice of semantics for the nominals, but our termination and completeness proofs later in the paper do not include global modalities. In the following let $\mathcal{H}$ denote a fixed finite Heyting algebra. That is, $\mathcal{H}$ is a finite lattice such that for all a and b in $\mathcal{H}$ there is a greatest element x of $\mathcal{H}$ satisfying $a \wedge x \leq b$. The element x is called the *relative pseudo-complement* of a with respect to b (denoted $a \Rightarrow b$). To avoid notational ambiguity in relation to the syntax of our hybrid logic, we will in the following use the symbol $\Rightarrow$ for relative pseudo-complement, and $\sqcup$ and $\sqcap$ for meet and join, respectively. The largest and smallest elements of $\mathcal{H}$ are denoted $\top$ and $\bot$, respectively. The elements of the Heyting algebra $\mathcal{H}$ are going to be used as truth values for our many-valued logic. Thus, in the following, we will often refer to the elements of $\mathcal{H}$ as *truth values*.[2]

2.2.1 Syntax for MVHL

Let a countable infinite set of propositional variables PROP and a countable infinite set of nominals NOM be given. In addition to the usual connectives of propositional modal logic, we include the global modalities E and A, and for every $i \in \mathsf{NOM}$, a satisfaction operator $@_i$.

Definition 1 (**MVHL**-formulas)**.** *The set of* ***MVHL****-formulas is given by the following grammar:*

$$\varphi \ ::= \ p \mid a \mid i \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \rightarrow \varphi) \mid \Box\varphi \mid \Diamond\varphi \mid @_i\varphi \mid E\varphi \mid A\varphi \ ,$$

[2]In order to give reasonable semantics for $\wedge$ and $\vee$ a Lattice structure is needed. A complete Lattice would be enough if the accessibility relation was only allowed to have two values, but since we also allows for the accessibility relation to take values in $\mathcal{H}$, the structure of a Heyting algebra is needed. For further discussions of the choice of a finite Heyting algebra as the set of truth values see [60, 61].

where $p \in \mathsf{PROP}$, $a \in \mathcal{H}$, *and* $i \in \mathsf{NOM}$.

In general we will use i, j, k and so on for nominals and a, b, c for elements of $\mathcal{H}$.

2.2.2 Semantics for MVHL

The semantics for **MVHL** is a Kripke semantics in which the accessibility relation is allowed to take values in $\mathcal{H}$. This is inspired by [61]. A model $\mathcal{M}$ is a tuple $\mathcal{M} = \langle W, R, \mathbf{n}, \nu \rangle$, where W is the set of worlds, and R a mapping $R : W \times W \to \mathcal{H}$ called the accessibility relation. $\mathbf{n}$ is a function interpreting the nominals, i.e. $\mathbf{n} : \mathsf{NOM} \to W$. Finally the valuation $\nu : W \times \mathsf{PROP} \to \mathcal{H}$ assigns truth values to the propositional variables at each world.

Now given a model $\mathcal{M} = \langle W, R, \mathbf{n}, \nu \rangle$, we can extend the valuation ν to all formulas in the following inductive way, where $w \in W$:

$$
\begin{array}{rcl}
\nu(w, a) & := & a \quad \text{for } a \in \mathcal{H} \\
\nu(w, i) & := & \begin{cases} \top & \text{, if } \mathbf{n}(i) = w \\ \bot & \text{, else} \end{cases} \\
\nu(w, \varphi \wedge \psi) & := & \nu(w, \varphi) \sqcap \nu(w, \psi) \\
\nu(w, \varphi \vee \psi) & := & \nu(w, \varphi) \sqcup \nu(w, \psi) \\
\nu(w, \varphi \to \psi) & := & \nu(w, \varphi) \Rightarrow \nu(w, \psi) \\
\nu(w, \Box\varphi) & := & \bigsqcap\{R(w, v) \Rightarrow \nu(v, \varphi) \mid v \in W\} \\
\nu(w, \Diamond\varphi) & := & \bigsqcup\{R(w, v) \sqcap \nu(v, \varphi) \mid v \in W\} \\
\nu(w, @_i\varphi) & := & \nu(\mathbf{n}(i), \varphi) \\
\nu(w, A\varphi) & := & \bigsqcap\{\nu(v, \varphi) \mid v \in W\} \\
\nu(w, E\varphi) & := & \bigsqcup\{\nu(v, \varphi) \mid v \in W\}
\end{array}
$$

The semantics chosen for the hybrid logical constructions is discussed in the following.[3] The semantics for $@_i\varphi$ is obvious, its truth value is simply the truth value of φ at the world i denotes. This is motived by the semantics of $@_i\varphi$ in standard hybrid logic. The semantics chosen for the global modalities A and E reflect the fact that these modalities are simply the global versions of

[3] In this chapter one particular semantic is chosen for the nominals and the satisfaction operators, however there are several other ways of defining the semantics of these hybrid logic constructs. Chapter 3 is dedicated to exploring other ways of defining many-valued semantics for these hybrid logic constructs.

the modalities $\Box$ and $\Diamond$. The choice of semantics for nominals is less obvious. In this paper we have chosen to assign each nominal i the value $\top$ in exactly one world, and $\bot$ in all other worlds. This is in agreement with the the standard semantics for hybrid logic in which a nominal "points to a unique world". It would probably also be possible to allow nominals to take values outside the set $\{\top, \bot\}$, but at least a nominal should receive the value $\top$ in one and only one world in order for the semantics to be in accordance with classical, two-valued, hybrid logic (and for nominals to be semantically different from ordinary propositional symbols). Our decision of making the semantics of nominals two-valued rests primarily on the fact that it allows us to preserve the following well-known logical equivalence from classical, two-valued, hybrid logic:

$$@_i\varphi \leftrightarrow E(i \wedge \varphi)$$
$$@_i\varphi \leftrightarrow A(i \rightarrow \varphi)$$

With the chosen semantics, these equivalences also hold in MVHL:

$$\nu(w, @_i\varphi) = \nu(\mathbf{n}(i), \varphi) = \bigsqcup\{\nu(v, i) \sqcap \nu(v, \varphi) \mid v \in W\} = \nu(w, E(i \wedge \varphi))$$
$$\nu(w, @_i\varphi) = \nu(\mathbf{n}(i), \varphi) = \bigsqcap\{\nu(v, i) \Rightarrow \nu(v, \varphi) \mid v \in W\} = \nu(w, A(i \rightarrow \varphi)).$$

Here we have been using that the following holds in a Heyting algebra: $\top \sqcap a = a$, $\bot \sqcap a = \bot$, $a \sqcup \bot = a$, $\top \Rightarrow a = a$ and $\bot \Rightarrow a = \top$. Another pleasant property resulting from the choice of semantics for nominals is the following:

$$\nu(w, @_i\Diamond j) = \nu(\mathbf{n}(i), \Diamond j) = \bigsqcup\{R(\mathbf{n}(i), v) \sqcap \nu(v, j) \mid v \in W\} = R(\mathbf{n}(i), \mathbf{n}(j)).$$

This identity expresses that the reachability of the world denoted by j from the world denoted by i is described by the formula $@_i\Diamond j$. This property also holds in classical hybrid logic. Identity between worlds denoted by nominals can also be expressed as usual, since we have:

$$\nu(w, @_i j) = \top \text{ iff } \mathbf{n}(i) = \mathbf{n}(j).$$

2.2.3 The relation to intuitionistic hybrid logic

As pointed out in the paper [60], there is a close relation between the many-valued modal logic given in that paper and intuitionistic modal logic. We shall in this subsection consider the relation between many-valued hybrid logic and a variant of the intuitionistic hybrid logic given in the paper [44] (which in turn is a hybridization of an intuitionistic modal logic introduced in a tense-logical version in [56]). In the present subsection we do not assume that a finite Heyting algebra has been fixed in advance, so the only atomic formulas we consider are ordinary propositional symbols, nominals, and the symbol $\perp$. We first define an appropriate notion of an intuitionistic model, which can be seen as a restricted variant of the notion of a model given in [44][4].

Definition 2. *A* restricted model *for intuitionistic hybrid logic is a tuple*

$$(W, \leq, D, \{R_w\}_{w \in W}, \{\nu_w\}_{w \in W})$$

where

1. *W is a non-empty finite set partially ordered by $\leq$;*
2. *D is a non-empty set;*
3. *for each w, R_w is a binary relation on D such that $w \leq v$ implies $R_w \subseteq R_v$; and*
4. *for each w, ν_w is a function that to each ordinary propositional symbol p assigns a subset of D such that $w \leq v$ implies $\nu_w(p) \subseteq \nu_v(p)$.*

Note that, the set D is to be understood as the set of possible worlds and is used to interpret the modal and hybrid part of the language (occasionally together with the set W). The elements of the set W are states of knowledge and for any such state w, the relation R_w is the set of known relationships between possible worlds and the set $\nu_w(p)$ is the set of possible worlds at which p is known to be true. Note that the definition requires that the epistemic partial order $\leq$ preserves these kinds of knowledge, that is, if an advance to a greater state of knowledge is made, then what is known is preserved.

Given a restricted model $\mathfrak{M} = (W, \leq, D, \{R_w\}_{w \in W}, \{\nu_w\}_{w \in W})$, an *assignment* is a function $\mathbf{n}$ that to each nominal assigns an element of D. The

[4]Compare to Definition 2, p. 237, of the paper [44]. The differences are the following: i) In [44] the set W need not be finite. ii) Instead of D there is a family $\{D_w\}_{w \in W}$ of non-empty sets such that $w \leq v$ implies $D_w \subseteq D_v$, R_w is a binary relation on D_w, and $\nu_w(p)$ is a subset of D_w. iii) There is a family $\{\sim_w\}_{w \in W}$ where $\sim_w$ is an equivalence relation on D_w such that $w \leq v$ implies $\sim_w \subseteq \sim_v$ and such that if $d \sim_w d'$, $e \sim_w e'$, and dR_we, then $d'R_we'$, and similarly, if $d \sim_w d'$ and $d \in \nu_w(p)$, then $d' \in \nu_w(p)$. The equivalence relations are used for the interpretation of nominals. Such a model for intuitionistic hybrid logic corresponds to a standard model for intuitionistic first-order logic with equality where equality is interpreted using the equivalence relations, cf. [148].

relation $\mathfrak{M}, \mathbf{n}, w, d \models \phi$ is defined by induction, where w is an element of W, $\mathbf{n}$ is an assignment, d is an element of D, and ϕ is a formula.

$$
\begin{array}{rll}
\mathfrak{M}, \mathbf{n}, w, d \models p & \text{iff} & d \in \nu_w(p) \\
\mathfrak{M}, \mathbf{n}, w, d \models i & \text{iff} & d = \mathbf{n}(i) \\
\mathfrak{M}, \mathbf{n}, w, d \models \phi \wedge \psi & \text{iff} & \mathfrak{M}, \mathbf{n}, w, d \models \phi \text{ and } \mathfrak{M}, \mathbf{n}, w, d \models \psi \\
\mathfrak{M}, \mathbf{n}, w, d \models \phi \vee \psi & \text{iff} & \mathfrak{M}, \mathbf{n}, w, d \models \phi \text{ or } \mathfrak{M}, \mathbf{n}, w, d \models \psi \\
\mathfrak{M}, \mathbf{n}, w, d \models \phi \rightarrow \psi & \text{iff} & \text{for all } v \geq w, \\
 & & \mathfrak{M}, \mathbf{n}, v, d \models \phi \text{ implies } \mathfrak{M}, \mathbf{n}, v, d \models \psi \\
\mathfrak{M}, \mathbf{n}, w, d \models \bot & \text{iff} & \text{falsum} \\
\mathfrak{M}, \mathbf{n}, w, d \models \Box\phi & \text{iff} & \text{for all } v \geq w, \text{ for all } e \in D, \\
 & & dR_v e \text{ implies } \mathfrak{M}, \mathbf{n}, v, e \models \phi \\
\mathfrak{M}, \mathbf{n}, w, d \models \Diamond\phi & \text{iff} & \text{for some } e \in D,\ dR_w e \text{ and } \mathfrak{M}, \mathbf{n}, w, e \models \phi \\
\mathfrak{M}, \mathbf{n}, w, d \models @_i\phi & \text{iff} & \mathfrak{M}, \mathbf{n}, w, \mathbf{n}(i) \models \phi \\
\mathfrak{M}, \mathbf{n}, w, d \models A\phi & \text{iff} & \text{for all } v \geq w, \text{ for all } e \in D,\ \mathfrak{M}, \mathbf{n}, v, e \models \phi \\
\mathfrak{M}, \mathbf{n}, w, d \models E\phi & \text{iff} & \text{for some } e \in D,\ \mathfrak{M}, \mathbf{n}, w, e \models \phi
\end{array}
$$

This semantics can be looked upon in two different ways: As indicated above, it can be seen as a restricted variant of the semantics given in [44], but it can also be seen as a hybridized version of a semantics given in the paper [60]. In the latter paper, the epistemic worlds of the semantics are thought of as experts and the epistemic partial order is thought of as a relation of dominance between experts: One expert dominates another one if whatever the first expert says is true is also said to be true by the second expert.

As pointed out in [60], the intuitionistic semantics for modal logic is in a certain sense equivalent to the many-valued semantics. This also holds in the hybrid-logical case. In what follows, we outline this equivalence. It can be shown that given a restricted model $\mathfrak{M} = (W, \leq, D, \{R_w\}_{w\in W}, \{\nu_w\}_{w\in W})$, cf. Definition 2, and an assignment $\mathbf{n}$, the $\leq$-closed subsets of W ordered by $\subseteq$ constitute a finite Heyting algebra, and moreover, a many-valued model $(D, R^*, \mathbf{n}, \nu^*)$ can be defined by letting

- $R^*(d, e) = \{w \in W \mid dR_w e\}$ and
- $\nu^*(d, p) = \{w \in W \mid d \in \nu_w(p)\}$.

By a straightforward extension of the corresponding proof in [60], it can be proved that for any formula ϕ, it is the case that $\nu^*(d, \phi) = \{w \in W \mid \mathfrak{M}, \mathbf{n}, w, d \models \phi\}$. Conversely, given a finite Heyting algebra $\mathcal{H}$ and a many-valued model $(D, R, \mathbf{n}, \nu)$, a restricted model $\mathfrak{M} = (W, \subseteq, D, \{R^*_w\}_{w\in W}, \{\nu^*_w\}_{w\in W})$ can be defined by letting

- $W = \{w \mid w \text{ is a proper prime filter in } \mathcal{H}\}$,
- $dR^*_w e$ if and only if $R(d,e) \in w$, and
- $d \in \nu^*_w(p)$ if and only if $\nu(d,p) \in w$.

Details can be found in the paper [60]. Again, by a straightforward extension of the corresponding proof in that paper, it can be proved that for any formula ϕ, it is the case that $\mathfrak{M}, \mathbf{n}, w, d \models \phi$ if and only if $\nu(d, \phi) \in w$.

Thus, in the above sense the intuitionistic semantics for hybrid logic is equivalent to the many-valued semantics for hybrid logic. It is an interesting question whether there is such an equivalence if instead of the restricted models of Definition 2 one considers the more general models for intuitionistic hybrid logic given in the paper [44][5]. We shall leave this to further work.

2.3 A tableau calculus for MVHL

In the following we will present a tableau calculus for MVHL. The basic notions for tableaux are defined as usual (see e.g. [58]). The formulas occurring in our tableaux will all be of the form $@_i(a \to \varphi)$ or $@_i(\varphi \to a)$ prefixed either a T or an F, where $i \in \mathsf{NOM}$ and $a \in \mathcal{H}$. That is, the formulas occurring in our tableaux will be *signed formulas* of hybrid logic. A signed formula of the form $T@_i(a \to \varphi)$ is used to express that the formula $a \to \varphi$ is true at i, that is, receives the value $\top$ at i. If $\nu(\mathbf{n}(i), a \to \varphi) = \top$ then, by definition of ν, $a \Rightarrow \nu(\mathbf{n}(i), \varphi) = \top$. By definition of relative pseudo-complement we then get that $\top$ is the greatest element of $\mathcal{H}$ satisfying $a \wedge \top \leq \nu(\mathbf{n}(i), \varphi)$. In other words, we simply have $a \leq \nu(\mathbf{n}(i), \varphi)$. Thus what is expressed by a formula $T@_i(a \to \varphi)$ is that the truth value of φ at i is greater than or equal to a. Symmetrically, a signed formula of the form $T@_i(\varphi \to a)$ expresses that the truth value of φ at i is less than or equal to a. Dually, a signed formula of the form $F@_i(a \to \varphi)$ ($F@_i(\varphi \to a)$) expresses that the truth value of φ at i is *not* greater than or equal to (less than or equal to) a.

The tableau rules are divided into four classes; Branch Closing Rules, Non-modal Rules, Modal Rules and Hybrid Rules. The Branch Closing Rules and Propositional Rules are direct translations of Fitting's corresponding rules for the pure modal case [61].

[5] As indicated in the previous footnote, in the intuitionistic semantics of [44], nominals are interpreted using a family $\{\sim_w\}_{w \in W}$ of equivalence relations, not identity. This seems to imply that in an equivalent many-valued semantics, nominals should be allowed to take on arbitrary truth-values, not just top and bottom.

Branch Closing Rules:

A tableau branch Θ is said to be *closed* if one of the following holds:

1. $T@_i(a \to b) \in \Theta$, for some a, b with $a \not\leq b$.
2. $F@_i(a \to b) \in \Theta$, for some a, b with $a \leq b$, $a \neq \bot$, and $b \neq \top$.
3. $F@_i(\bot \to \varphi) \in \Theta$, for some formula φ.
4. $F@_i(\varphi \to \top) \in \Theta$, for some formula φ.
5. $T@_i(b \to \varphi), F@_i(a \to \varphi) \in \Theta$, for some a, b with $a \leq b$.
6. $T@_j(a \to i), F@_i(b \to j) \in \Theta$, for some $a, b \neq \bot$.
7. $T@_i(i \to a) \in \Theta$, for some nominal i and truth value a with $a \neq \top$.

The last two conditions, 6 and 7, have no counterpart in Fitting's system, but are required in ours to deal with the semantics chosen for nominals. Note that if a formula $F@_i(a \to i)$ with $a \neq \top$ occurs on a branch then the branch can also be closed: In case $a = \bot$, condition 3 immediately implies closure. If $a \neq \bot$ then using the reversal rule ($\mathbf{F} \geq$) (see below), we can add a formula $T@_i(i \to b)$ to the branch, where b is one of the maximal members of $\mathcal{H}$ not above a. Because b is not above a, b cannot be $\top$. Thus condition 7 implies closure.

Non-modal Rules:

The tableau rules for the propositional connectives and the rules capturing the properties of the Heyting algebra are given in Figure 2.1 and Figure 2.2, respectively. The rules of Figure 2.2 are called *reversal rules*, as in [61]. The reversal rules together with the closure rules ensure that no formula can be assigned more than one truth value (relative to a given world and a given branch).

$$\frac{T@_i(a \to (\varphi \wedge \psi))}{\begin{array}{c} T@_i(a \to \varphi) \\ T@_i(a \to \psi) \end{array}}\ (\mathbf{T}\wedge)^1 \qquad \frac{F@_i(a \to (\varphi \wedge \psi))}{F@_i(a \to \varphi) \mid F@_i(a \to \psi)}\ (\mathbf{F}\wedge)^1$$

$$\frac{T@_i((\varphi \vee \psi) \to a)}{\begin{array}{c} T@_i(\varphi \to a) \\ T@_i(\psi \to a) \end{array}}\ (\mathbf{T}\vee)^2 \qquad \frac{F@_i((\varphi \vee \psi) \to a)}{F@_i(\varphi \to a) \mid F@_i(\psi \to a)}\ (\mathbf{F}\vee)^2$$

$$\frac{F@_i(a \to (\varphi \to \psi))}{\begin{array}{c|c|c} T@_i(b_1 \to \varphi) & \cdots & T@_i(b_n \to \varphi) \\ F@_i(b_1 \to \psi) & \cdots & F@_i(b_n \to \psi) \end{array}}\ (\mathbf{F}\to)^3 \qquad \frac{T@_i(a \to (\varphi \to \psi))}{F@_i(b \to \varphi) \mid T@_i(b \to \psi)}\ (\mathbf{T}\to)^4$$

[1] Where $a \neq \bot$.
[2] Where $a \neq \top$.
[3] Where $a \neq \bot$ and $b_1, ..., b_n$ are all the members of $\mathcal{H}$ with $b_i \leq a$ except $\bot$.
[4] Where $a \neq \bot$ and b is any member of $\mathcal{H}$ with $b \leq a$ except $\bot$.

Figure 2.1: Propositional Rules for **MVHL**.

$$\frac{F@_i(a \to \varphi)}{T@_i(\varphi \to b_1) \mid \cdots \mid T@_i(\varphi \to b_n)}\ (\mathbf{F}\geq)^{1,2} \qquad \frac{T@_i(a \to \varphi)}{F@_i(\varphi \to b)}\ (\mathbf{T}\geq)^{1,3}$$

$$\frac{F@_i(\varphi \to a)}{T@_i(b_1 \to \varphi) \mid \cdots \mid T@_i(b_n \to \varphi)}\ (\mathbf{F}\leq)^{1,4} \qquad \frac{T@_i(\varphi \to a)}{F@_i(b \to \varphi)}\ (\mathbf{T}\leq)^{1,5}$$

[1] φ is a formula other than a propositional constant from $\mathcal{H}$.
[2] Where $b_1, ..., b_n$ are all maximal members of $\mathcal{H}$ with $a \nleq b_i$ and $a \neq \bot$.
[3] Where b is any maximal member of $\mathcal{H}$ with $a \nleq b$ and $a \neq \bot$.
[4] Where $b_1, ..., b_n$ are all minimal members of $\mathcal{H}$ with $b_i \nleq a$ and $a \neq \top$.
[5] Where b is any minimal member of $\mathcal{H}$ with $b \nleq a$ and $a \neq \top$.

Figure 2.2: Reversal Rules for **MVHL**.

Modal Rules:

These modal rules, presented in Figure 2.3, differ from the ones of Fitting and heavily employs the hybrid logic machinery.[6] Note that the tableau rules contain formulas of the form $T@_i(a \leftrightarrow \Diamond j)$. Such formulas are simply used as shorthand notation for the occurrence of both the formulas $T@_i(a \to \Diamond j)$ and $T@_i(\Diamond j \to a)$. In each of the rules of our calculus, the leftmost premise is called the *principal premise*. If α is a signed formula of one of the forms $T@_i(a \to \varphi)$, $T@_i(\varphi \to a)$, $F@_i(a \to \varphi)$ or $F@_i(\varphi \to a)$, we call φ the *body* of α and i its *prefix*. If α and β are two signed formulas such that the body of α is a subformula of the body of β, then α is said to be a *quasi-subformula* of β.

Hybrid Rules:

These hybrid rules, presented in Figure 2.4, are inspired by the standard rules from classical hybrid logic (see [25, 39, 37]).[7] Note that for the **(NOM)** rule, two versions are needed. Furthermore a new rule is needed due to the fact that we are in a many-valued setting, this is the rule **(NOM EQ)**, which ensures our semantic definition of nominals as being $\top$ in exactly one world.

A *tableau proof* of a formula ϕ is a closed tableau with root $F@_i(\top \to \phi)$, where i is an arbitrary nominal not occurring in ϕ. The intuition here is that the root formula $F@_i(\top \to \phi)$ asserts that ϕ does *not* have the value $\top$, and if the tableau closes, this assertion is refuted. If i is a nominal occurring in the root formula of a tableau then i is called a *root nominal* of the tableau. Other nominals occurring on the tableau are called *non-root nominals*.

[6]In [61] the modal tableau rules are so called destructive rules (see [62]) which replaces an entire branch of a tableau with new branch. The modal rules given here is standard modal rules that simply add new formulas to the end of an existing branch or split the branch into new branches ending with new formulas. Thus rules of Figure 2.3 are not easily comparable to the modal rules of [61]. The rules of Figure 2.3 are inspired by the modal rules in Figure 1.8 of Section 1.2.2. The $(\Diamond)$ rule of Figure 1.8 is comparable to the $(\mathbf{F}\Diamond)$ rule and the $(\neg\Diamond)$ rule of Figure 1.8 is comparable to the $(\mathbf{T}\Diamond)$ rule.

[7]Again, compare to the rules of Figure 1.8 of Section 1.2.2. Moving from two-valued to many-valued hybrid logic has changed the rules quite a lot, however the bridge rules are comparable and the **(F-NOM)** and **(T-NOM)** are comparable to the **(NOM1)** rule of Figure 1.8.

$$\dfrac{F@_i(a \to \Box\varphi)}{\begin{array}{c|c|c} T@_i(b_1 \leftrightarrow \Diamond j) & \cdots & T@_i(b_n \leftrightarrow \Diamond j) \\ F@_j((a \sqcap b_1) \to \varphi) & \cdots & F@_j((a \sqcap b_n) \to \varphi) \end{array}}\ \mathbf{(F\Box)}^1$$

$$\dfrac{T@_i(a \to \Box\varphi) \qquad T@_i(b \to \Diamond j)}{T@_j((a \sqcap b) \to \varphi)}\ \mathbf{(T\Box)}$$

$$\dfrac{F@_i(\Diamond\varphi \to a)}{\begin{array}{c|c|c} T@_i(b_1 \leftrightarrow \Diamond j) & \cdots & T@_i(b_n \leftrightarrow \Diamond j) \\ F@_j(\varphi \to (b_1 \Rightarrow a)) & \cdots & F@_j(\varphi \to (b_n \Rightarrow a)) \end{array}}\ \mathbf{(F\Diamond)}^{1,2}$$

$$\dfrac{T@_i(\Diamond\varphi \to a) \qquad T@_i(b \to \Diamond j)}{T@_j(\varphi \to (b \Rightarrow a))}\ \mathbf{(T\Diamond)}^2$$

$$\dfrac{F@_i(E\varphi \to a)}{F@_j(\varphi \to a)}\ \mathbf{(FE)}^3 \qquad \dfrac{T@_i(E\varphi \to a)}{T@_j(\varphi \to a)}\ \mathbf{(TE)}^4$$

$$\dfrac{T@_i(a \to A\varphi)}{T@_j(a \to \varphi)}\ \mathbf{(TA)}^4 \qquad \dfrac{F@_i(a \to A\varphi)}{F@_j(a \to \varphi)}\ \mathbf{(FA)}^3$$

[1] Where $\mathcal{H} = \{b_1, ..., b_n\}$ and j is a nominal new to the branch.
[2] Where the principal premise is a quasi-subformula of the root formula.
[3] Where j is a nominal new to the branch.
[4] Where j is a nominal already occurring on the branch.

Figure 2.3: Modal Rules for **MVHL**.

$$\frac{T@_i(@_j\varphi \to a)}{T@_j(\varphi \to a)}\,(@_L) \qquad \frac{T@_i(a \to @_j\varphi)}{T@_j(a \to \varphi)}\,(@_R)$$

$$\frac{F@_i\varphi \qquad T@_i(a \to j)}{F@_j\varphi}\,(\textbf{F-NOM})^{1,2} \qquad \frac{T@_i\varphi \qquad T@_i(a \to j)}{T@_j\varphi}\,(\textbf{T-NOM})^{1,2}$$

$$\frac{T@_k(\Diamond i \to b) \qquad T@_i(a \to j)}{T@_k(\Diamond j \to b)}\,(\textbf{BRIDGE}_L)^1 \qquad \frac{T@_k(b \to \Diamond i) \qquad T@_i(a \to j)}{T@_k(b \to \Diamond j)}\,(\textbf{BRIDGE}_R)^1$$

$$\frac{T@_i(\top \to j) \qquad T@_j(\top \to k)}{T@_i(\top \to k)}\,(\textbf{TRANS})$$

$$\frac{T@_i(a \to j)}{T@_i(\top \to j)}\,(\textbf{NOM EQ})^1$$

[1] Where $a \neq \bot$.
[2] Where the principal premise is a quasi-subformula of the root formula.

Figure 2.4: Hybrid Rules for **MVHL**.

2.4 Termination

The tableau calculus presented above is not terminating. This is due to the rules (**TA**) and (**FA**) for the global modality A. If the rules for the global modalities—(**FE**), (**TE**), (**TA**) and (**FA**)—are all removed, we obtain a tableau calculus for the many-valued hybrid logic with these modalities removed. We will refer to this calculus as the *basic calculus*, and refer to its tableaux as *basic tableaux*. In the following we will prove that the basic calculus terminates. The proof closely follows the method introduced in [37] and sketched at the end of Section 1.2.2.

If α and β are signed formulas on a tableau branch, then β is said to be *produced* by α if β is one of the conclusions of a rule application with principal premise α. The signed formula β is said to be *indirectly produced* by α if there exists a sequence of signed formulas $\alpha, \alpha_1, \alpha_2, \ldots, \alpha_n, \beta$ in which each formula is produced by its predecessor. We now have the following result.

Lemma 3 (Quasi-subformula Property). *Let $\mathcal{T}$ be a basic tableau. For any signed formula α occurring on $\mathcal{T}$, one of the following holds:*

1. α is a quasi-subformula of the root formula of $\mathcal{T}$.

2. α is a formula of one of the forms $T@_i(a \to \Diamond j)$, $T@_i(\Diamond j \to a)$, $F@_i(a \to \Diamond j)$ or $F@_i(\Diamond j \to a)$, for which one of the following holds:

(a) j is a root nominal.

*(b) α is indirectly produced by (**F**$\Box$) or (**F**$\Diamond$) by a number of applications of the reversal rules.*

Proof. The proof goes by induction on the construction of $\mathcal{T}$. In the basic case α is just the root formula, which of course is of type 1. Now assume that α has been introduced by one of the propositional rules. These rules does not take premises of type 2 and thus by induction they must be of type 1. But then the conclusions produced by these rules must also be of type 1, thus α must be of type 1. If α has been produced by one of the reversal rules by a formula of type 1, then α will also be of type 1 and if α is produced by a formula of type 2, α is also of type 2. Now the modal rules. If α has been produced by the rule (**T**$\Box$) then the principal premise can not be a formula of type 2 and thus by induction it must be of type 1. But then so is α. Similar for the rule (**T**$\Diamond$) where the side condition insures that the principal premise is of type 1. If α is introduced by on of the rules (**F**$\Box$) or (**F**$\Diamond$) again the premise must be of type 1. These rules produce two formulas, the first one is by definition of type 2b and the second must be of type 1 since the premise is. Thus in this case α is either of type 1 or type 2b. Finally for the hybrid rules. In the rules (**TRANS**), (**NOM EQ**), ($@_L$) or ($@_R$) the premises can not be of type 2 and thus by induction they must be of type 1. But then the conclusions will also be of type 1. Now if the rule used is (**T-NOM**) or (**F-NOM**) then the side condition insures that the principal premise are of type 1. But then the conclusion will also be of type 1. Now assume that one of the rules (**BRIDGE**$_L$) or (**BRIDGE**$_R$) have been applied to produce α. Then the non-principal premise can not be of type 2 and thus must be of type 1, which implies that j is a root nominal. Thus the conclusion α must be of type 2a. This completes the proof. $\Box$

Note that in the basic calculus the only rules that can introduce new nominals to a tableau are (**F**$\Box$) and (**F**$\Diamond$).

Definition 4. *Let Θ be a branch of a basic tableau. If a nominal j has been introduced to the branch by applying either (**F**$\Box$) or (**F**$\Diamond$) to a premise with prefix i then we say that j is* generated *by i on Θ, and we write $i \prec_\Theta j$.*

Lemma 5. *Let Θ be a branch of a basic tableau. The graph $G = (N^\Theta, \prec_\Theta)$, where N^Θ is the set of nominals occurring on Θ, is a finite set of wellfounded, finitely branching trees.*

Proof. That G is wellfounded follows from the observation that if $i \prec_\Theta j$, then the first occurrence of i on Θ is before the first occurrence of j. That G is finitely branching is shown as follows. For any given nominal i the number of nominals j satisfying $i \prec_\Theta j$ is bounded by the number of applications of $(\mathbf{F}\Box)$ and $(\mathbf{F}\Diamond)$ to premises of the form $F@_i(a \to \Box\varphi)$ and $F@_i(\Diamond\varphi \to a)$. So to prove that G is finitely branching, we only need to prove that for any given i the number of such premises is finite. However, this follows immediately from the fact that all such premises must be quasi-subformulas of the root formula (cf. Lemma 3 and the condition on applications of $(\mathbf{F}\Diamond)$). What is left is to prove that G is a finite set of trees. This follows from the fact that each nominal in N^Θ can be generated by at most one other nominal, and the fact that each nominal in N^Θ must have one of the finitely many root nominals of Θ as an ancestor. $\Box$

Lemma 6. *Let Θ be a branch of a basic tableau. Then Θ is infinite if and only if there exists an infinite chain of nominals*

$$i_1 \prec_\Theta i_2 \prec_\Theta i_3 \prec_\Theta \cdots .$$

Proof. The 'if' direction is trivial. To prove the 'only if' direction, let Θ be any infinite tableau branch. Θ must contain infinitely many distinct nominals, since it follows immediately from Lemma 3 that a tableau with finitely many nominals can only contain finitely many distinct formulas. This implies that the graph $G = (N^\Theta, \prec_\Theta)$ defined as in Lemma 5 must be infinite. Since by Lemma 5, G is a finite set of wellfounded, finitely branching trees, G must then contain an infinite path $(i_1, i_2, i_3, \dots)$. Thus we get an infinite chain $i_1 \prec_\Theta i_2 \prec_\Theta i_3 \prec_\Theta \cdots$. $\Box$

Definition 7. *Let Θ be a branch of a basic tableau, and let i be a nominal occurring on Θ. We define $m_\Theta(i)$ to be the maximal length of any formula with prefix i occurring on Θ.*

Lemma 8 (Decreasing length). *Let Θ be a branch of a basic tableau. If $i \prec_\Theta j$ then $m_\Theta(i) > m_\Theta(j)$.*

Proof. For any signed formula α, we will use $|\alpha|$ to denote the length of α. Assume $i \prec_\Theta j$. Let α be a signed formula satisfying: 1) α has maximal length among the formulas on Θ with prefix j; 2) α is the earliest occurring formula on Θ with this property. We need to prove $m_\Theta(i) > |\alpha|$. The formula α can not have been introduced on Θ by applying any of the propositional rules (Figure 2.1), since this would contradict maximality of α. It can not have been directly produced by any of the reversal rules (Figure 2.2) either, since this would contradict the choice of α as the earliest possible on Θ of maximal length with prefix j. By the same argument, α can not have been directly produced by any of the rules (**BRIDGE**$_L$), (**BRIDGE**$_R$), (**TRANS**) or (**NOM EQ**). Assume now α has been introduced by applying ($@_L$) or ($@_R$) to a premise of the form $T@_k(@_j\varphi \to a)$ or $T@_k(a \to @_j\varphi)$. By Lemma 3, the premise must be a quasi-subformula of the root formula. Thus j must be a root nominal. However, this is a contradiction, since by assumption j is generated by i, and can thus not be a root nominal. Thus neither ($@_L$) nor ($@_R$) can have been the rule producing α. Now assume that α has been produced by an application of either (**F-NOM**) or (**T-NOM**). Since α has index j, the non-principal premise used in this rule application must have the form $T@_i(a \to j)$. By Lemma 3, this premise must be a quasi-subformula of the root formula, and thus j is again a root nominal, which is a contradiction. Thus α can not have been produced by (**F-NOM**) or (**T-NOM**) either. Thus α must have been introduced by one of the rules ($\mathbf{F}\Box$), ($\mathbf{T}\Box$), ($\mathbf{F}\Diamond$) or ($\mathbf{T}\Diamond$). Consider first the case of the ($\mathbf{F}\Box$) and ($\mathbf{F}\Diamond$) rules. If an instance of one of these produced α, then this instance must have been applied to a premise β with prefix i, since we have assumed $i \prec_\Theta j$ and by Lemma 5 there cannot be an $i' \neq i$ satisfying $i' \prec_\Theta j$. (Note that if α is of the form $T@_j(b \to \Diamond k)$ or $T@_j(\Diamond k \to b)$ produced by a formula $F@_j(a \to \Box\varphi)$ or $F@_j(\Diamond\varphi \to a)$, this would lead to a contradiction with the assumption that α has maximal length with prefix j and is the earliest occurring formula with this property.) Since the rules in question always produce conclusions that are shorter than their premises, β must be longer than α. Since β is a formula with prefix i we then get:

$$m_\Theta(i) \geq |\beta| > |\alpha|\,, \tag{2.1}$$

as required. Finally, consider the case where α has been produced by either ($\mathbf{T}\Box$) or ($\mathbf{T}\Diamond$). Then α has been produced by a rule instance with non-principal premise of the form $T@_k(b \to \Diamond j)$. Since j is not a root nominal, this premise can not be a quasi-subformula of the root formula. Neither can it be of the tybe ($2a$) mentioned in Lemma 3. It must thus be of type ($2b$), that is, it must be indirectly produced by formulas of the form $T@_k(b_m \to \Diamond j')$

or $T@_k(\Diamond j' \to b_m)$ obtained as conclusion by applications of **(F$\Box$)** or **(F$\Diamond$)**. Since only reversal rules have been applied in the indirect production from these conclusions, we must have $j = j'$ and thus $k \prec_\Theta j$. Since we already have $i \prec_\Theta j$ we get $k = i$, using Lemma 5. We can conclude that the non-principal premise of the rule instance producing α must have the form $T@_i(b \to \Diamond j)$, and thus the principal premise must be a formula β with index i. Since the rules in question always produce conclusions that are shorter than their premises, β must be longer than α. Since β is a formula with prefix i we then again get the sequence of inequalities (2.1), as required. $\square$

We can now finally prove termination of the basic calculus.

Theorem 9 (Termination of the basic calculus). *Any tableau in the basic calculus is finite.*

Proof. Assume there exists an infinite basic tableau. Then it must have an infinite branch Θ. By Lemma 6, there exists an infinite chain

$$i_1 \prec_\Theta i_2 \prec_\Theta i_3 \prec_\Theta \cdots .$$

Now by Lemma 8 we have

$$m_\Theta(i_1) > m_\Theta(i_2) > m_\Theta(i_3) > \cdots$$

which is a contradiction, since $m_\Theta(i)$ is a non-negative number for any nominal i. $\square$

2.5 Completeness of the basic calculus

In this section we prove completeness of the basic calculus, that is, the calculus without the global modalities. However, we remark that one can prove completeness for a calculus including the global modalities similar to the calculus of the present paper. Let Θ be an open saturated branch in the tableau calculus. We will use this branch to construct a model $\mathcal{M}_\Theta = \langle W_\Theta, R_\Theta, \mathbf{n}_\Theta, \nu_\Theta \rangle$. The set of worlds, W_Θ is simply defined to be the set of nominals occurring on Θ. The definition of the other elements of the model requires a bit more work. First we define the mapping $\mathbf{n}_\Theta$.

Fix a choice function σ that for any given set of nominals on Θ returns one of these nominals. We now define the mapping $\mathbf{n}_\Theta$ in the following way:

$$\mathbf{n}_\Theta(i) = \begin{cases} \sigma\{j \mid T@_i(\top \to j) \in \Theta\} & \text{if } \{j \mid T@_i(\top \to j) \in \Theta\} \neq \emptyset \\ i & \text{otherwise.} \end{cases}$$

A nominal i is called an *urfather* on Θ if $i = \mathbf{n}_\Theta(j)$ for some nominal j.

Lemma 10. *Let Θ be a saturated tableau branch. Then we have the following properties:*

1. *If $T@_i\varphi \in \Theta$ is a quasi-subformula of the root formula then $T@_{\mathbf{n}_\Theta(i)}\varphi \in \Theta$. Similarly, if $F@_i\varphi \in \Theta$ is a quasi-subformula of the root formula then $F@_{\mathbf{n}_\Theta(i)}\varphi \in \Theta$.*
2. *If $T@_i(\top \to j) \in \Theta$ then $\mathbf{n}_\Theta(i) = \mathbf{n}_\Theta(j)$.*
3. *If i is an urfather on Θ then $\mathbf{n}_\Theta(i) = i$.*

Proof. First we prove (i). Assume $T@_i\varphi \in \Theta$ is a quasi-subformula of the root formula. If $\mathbf{n}_\Theta(i) = i$ then there is nothing to prove. So assume $\mathbf{n}_\Theta(i) = \sigma\{j \mid T@_i(\top \to j) \in \Theta\}$. Then $T@_i(\top \to \mathbf{n}_\Theta(i)) \in \Theta$, and by applying (**T-NOM**) to premises $T@_i\varphi$ and $T@_i(\top \to \mathbf{n}_\Theta(i))$ we get $T@_{\mathbf{n}_\Theta(i)}\varphi$, as needed. The case of $F@_i\varphi \in \Theta$ is proved similarly, using (**F-NOM**) instead of (**T-NOM**). We now prove (ii). Assume $T@_i(\top \to j) \in \Theta$. To prove $\mathbf{n}_\Theta(i) = \mathbf{n}_\Theta(j)$ it suffices to prove that for all nominals k, $T@_i(\top \to k) \in \Theta \Leftrightarrow T@_j(\top \to k) \in \Theta$. So let k be an arbitrary nominal. If $T@_i(\top \to k) \in \Theta$ then we can apply (**T-NOM**) (since $T@_i(\top \to k)$ is a quasi-subformula of the root formula by Lemma 3) to premises $T@_i(\top \to k)$ and $T@_i(\top \to j)$ to obtain the conclusion $T@_j(\top \to k)$, as required. If conversely $T@_j(\top \to k) \in \Theta$ then we can apply (**TRANS**) to premises $T@_i(\top \to j)$ and $T@_j(\top \to k)$ to obtain the conclusion $T@_i(\top \to k)$, as required. We finally prove (iii). Assume i is an urfather. Then $i = \mathbf{n}_\Theta(j)$ for some j. If $j = i$ we are done. Otherwise we have $i = \mathbf{n}_\Theta(j) = \sigma\{k \mid T@_j(\top \to k) \in \Theta\}$ and thus $T@_j(\top \to i) \in \Theta$. This implies $i = \mathbf{n}_\Theta(j) = \mathbf{n}_\Theta(i)$, using item (ii). ☐

We now turn to the definition of ν_Θ. As in [61] we will not define a particular valuation ν of the propositional variables occuring on the branch, but only show that any valuation assigning values between a certain lower and upper bound (both given by the branch Θ) will do. Let us first define these bounds.

Definition 11. *For a formula φ in the language of MVHL and a nominal i, define:*

$$bound^{\Theta,i}(\varphi) = \bigsqcap\{a \mid T@_i(\varphi \to a) \in \Theta\}$$
$$bound_{\Theta,i}(\varphi) = \bigsqcup\{a \mid T@_i(a \to \varphi) \in \Theta\}$$

The intuition is that $bound^{\Theta,i}(\varphi)$ is an upper bound for the truth value of φ at the world i decided by the branch Θ and $bound_{\Theta,i}(\varphi)$ is a lower bound for this truth value.

The following lemma corresponds to Lemma 6.4 of [61] and can be proved in the same way. It ensures that we can actually always chose a value between the lower and the upper bounds.

Lemma 12. *For all i on Θ and all formulas φ of MVHL*

$$bound_{\Theta,i}(\varphi) \leq bound^{\Theta,i}(\varphi).$$

Later we will show that any valuation assigning a value to p between $bound_{\Theta,i}(p)$ and $bound^{\Theta,i}(p)$ at the world $\mathbf{n}_\Theta(i)$ will do for the truth value of p at this world.

The following lemma corresponds to Proposition 6.5 in [61] and is proven in the same way.

Lemma 13. *Let φ be any formula in the MVHL language other than a propositional constant from $\mathcal{H}$, and let $a \in \mathcal{H}$, then:*

- *(i) If $T@_i(a \to \varphi) \in \Theta$, then $a \leq bound_{\Theta,i}(\varphi)$.*
- *(ii) If $T@_i(\varphi \to a) \in \Theta$, then $bound^{\Theta,i}(\varphi) \leq a$.*
- *(iii) If $F@_i(a \to \varphi) \in \Theta$, then $a \not\leq bound^{\Theta,i}(\varphi)$.*
- *(iv) If $F@_i(\varphi \to a) \in \Theta$, then $bound_{\Theta,i}(\varphi) \not\leq a$.*

The accessibility relation R_Θ is defined as follows:

$$R_\Theta(i,j) = \bigsqcup\{b \mid T@_i(b \to \Diamond k) \in \Theta, \mathbf{n}_\Theta(k) = j\}.$$

We have the following result, which we are going to use in proving completeness.

Lemma 14. *If $T@_i(c \leftrightarrow \Diamond j) \in \Theta$ then $R_\Theta(i, \mathbf{n}_\Theta(j)) = c$.*

Proof. We will prove $R_\Theta(i, \mathbf{n}_\Theta(j)) \geq c$ and $R_\Theta(i, \mathbf{n}_\Theta(j)) \leq c$. First we prove $R_\Theta(i, \mathbf{n}_\Theta(j)) \geq c$. Since $T@_i(c \leftrightarrow \Diamond j) \in \Theta$ we have $T@_i(c \to \Diamond j) \in \Theta$, and thus

$$\begin{aligned} R_\Theta(i, \mathbf{n}_\Theta(j)) &= \bigsqcup\{b \mid T@_i(b \to \Diamond k) \in \Theta, \mathbf{n}_\Theta(k) = \mathbf{n}_\Theta(j)\} \\ &\geq \bigsqcup\{b \mid T@_i(b \to \Diamond j) \in \Theta\} \\ &\geq c. \end{aligned}$$

We now prove $R_\Theta(i, \mathbf{n}_\Theta(j)) \leq c$. By definition of $\mathbf{n}_\Theta$ we have either $\mathbf{n}_\Theta(j) = j$ or $T@_j(\top \to \mathbf{n}_\Theta(j)) \in \Theta$. If $T@_j(\top \to \mathbf{n}_\Theta(j)) \in \Theta$ then since $T@_i(\Diamond j \to c) \in \Theta$ we get $T@_i(\Diamond \mathbf{n}_\Theta(j) \to c) \in \Theta$, using ($\mathbf{BRIDGE}_L$). If $\mathbf{n}_\Theta(j) = j$ we obviously also have $T@_i(\Diamond \mathbf{n}_\Theta(j) \to c) \in \Theta$. Applying Lemma 13 (ii) we then get $bound^{\Theta,i}(\Diamond \mathbf{n}_\Theta(j)) \leq c$. Thus

$$\begin{aligned}
R_\Theta(i, \mathbf{n}_\Theta(j)) &= \bigsqcup\{b \mid T@_i(b \to \Diamond k) \in \Theta, \mathbf{n}_\Theta(k) = \mathbf{n}_\Theta(j)\} \\
&\leq \bigsqcup\{b \mid T@_i(b \to \Diamond \mathbf{n}_\Theta(j)) \in \Theta\} \quad \text{(using } (\mathbf{BRIDGE}_R)) \\
&= bound_{\Theta,i}(\Diamond \mathbf{n}_\Theta(j)) \\
&\leq bound^{\Theta,i}(\Diamond \mathbf{n}_\Theta(j)) \quad \text{(using Lemma 12)} \\
&\leq c,
\end{aligned}$$

as required. □

The theorem we need for completeness now may be stated in the following way:

Theorem 15. *Let ν be a valuation such that for all propositional variables p and all urfather nominals i*

$$bound_{\Theta,i}(p) \leq \nu(i, p) \leq bound^{\Theta,i}(p).$$

Then for all subformulas φ of the body of root formula of Θ

$$bound_{\Theta,i}(\varphi) \leq \nu(i, \varphi) \leq bound^{\Theta,i}(\varphi).$$

Proof. By induction on φ. The base cases are where φ is a propositional variable p, a value $c \in \mathcal{H}$ or a nominal j. The case where φ is p follows directly by the assumption. The case where φ is c is easy: First note that for any truth values a, b, if $T@_i(a \to b) \in \Theta$ then $a \leq b$. This follows from closure rule 1 presented in Section 2.3. Thus we get:

$$bound_{\Theta,i}(c) = \bigsqcup\{a \mid T@_i(a \to c) \in \Theta\} \leq c \leq \bigsqcap\{a \mid T@_i(c \to a) \in \Theta\} = bound^{\Theta,i}(c).$$

Now assume φ is a nominal j. By definition of ν, $\nu(i, j)$ is $\top$ if $\mathbf{n}_\Theta(j) = i$ and $\bot$ otherwise. Assume first $\mathbf{n}_\Theta(j) = i$. Then $\nu(i, j)$ is $\top$, so trivially we have $bound_{\Theta,i}(j) \leq \nu(i, j)$. We thus only need to prove $\nu(i, j) \leq bound^{\Theta,i}(j)$, that is, we need to prove $\top = bound^{\Theta,i}(j) = \bigsqcap\{a \mid T@_i(j \to a) \in \Theta\}$. This amounts to showing that, for all $a \in \mathcal{H}$, $T@_i(j \to a) \in \Theta$ implies $a = \top$. Assume towards a contradiction that, for some a, $T@_i(j \to a) \in \Theta$ and $a \neq \top$. Since we have assumed $\mathbf{n}_\Theta(j) = i$, by definition of $\mathbf{n}_\Theta$ we get that either $j = i$

or $T@_j(\top \to i) \in \Theta$. If $j = i$ then we have that Θ contains a formula of the form $T@_i(i \to a)$ where $a \neq \top$. This immediately contradicts closure rule 7. Assume instead $T@_j(\top \to i) \in \Theta$. Since we also have $T@_i(j \to a) \in \Theta$ where $a \neq \top$, we can apply $(\mathbf{T} \leq)$ to conclude that that Θ must contain a formula of the form $F@_i(t \to j)$ where t is some truth value different from $\perp$. Since Θ then contains both $T@_j(\top \to i)$ and $F@_i(t \to j)$ where $t \neq \perp$, we get a contradiction by closure rule 6. Assume now $\mathbf{n}_\Theta(j) \neq i$. Then $\nu(i, j) = \perp$, and the inequality $\nu(i, j) \leq bound^{\Theta,i}(j)$ thus holds trivially. To prove the other inequality, $bound_{\Theta,i}(j) \leq \nu(i, j)$, we need to show that if $T@_i(a \to j) \in \Theta$ then $a = \perp$. Thus assume toward a contradiction that $T@_i(a \to j) \in \Theta$ and $a \neq \perp$. Then rule (**NOM EQ**) implies $T@_i(\top \to j) \in \Theta$. Thus, by item 2 of Lemma 10, we get $\mathbf{n}_\Theta(i) = \mathbf{n}_\Theta(j)$. Since i is assumed to be an urfather, item 3 of Lemma 10 implies $\mathbf{n}_\Theta(i) = i$. Thus we get $\mathbf{n}_\Theta(j) = \mathbf{n}_\Theta(i) = i$, contradiction the assumption.

Now for the induction step. First the case where φ is $@_j\psi$: Note that $\nu(i, @_j\psi) = \nu(\mathbf{n}_\Theta(j), \psi)$ and by induction hypothesis, since $\mathbf{n}_\Theta(j)$ is an urfather,

$$bound_{\Theta,\mathbf{n}_\Theta(j)}(\psi) \leq \nu(\mathbf{n}_\Theta(j), \psi) \leq bound^{\Theta,\mathbf{n}_\Theta(j)}(\psi).$$

Now by the rule $(\mathbf{@}_R)$, if $T@_i(a \to @_j\psi) \in \Theta$ then $T@_j(a \to \psi) \in \Theta$, for all $a \in \mathcal{H}$. Thus we get that

$$\begin{array}{rcll}
bound_{\Theta,i}(@_j\psi) & = & \bigsqcup\{a \mid T@_i(a \to @_j\psi) \in \Theta\} & \\
 & \leq & \bigsqcup\{a \mid T@_j(a \to \psi) \in \Theta\} & \\
 & \leq & \bigsqcup\{a \mid T@_{\mathbf{n}_\Theta(j)}(a \to \psi) \in \Theta\} & \text{(using 1 of Lemma 10)} \\
 & = & bound_{\Theta,\mathbf{n}_\Theta(j)}(\psi) & \\
 & \leq & \nu(\mathbf{n}_\Theta(j), \psi) & \\
 & = & \nu(i, @_j\psi). &
\end{array}$$

Similar by the $(\mathbf{@}_L)$ rule, $T@_i(@_j\psi \to a) \in \Theta$ implies that $T@_j(\psi \to a) \in \Theta$, for all $a \in \mathcal{H}$. Hence

$$\begin{aligned}
\nu(i, @_j\psi) &= \nu(\mathbf{n}_\Theta(j), \psi) \\
&\leq \mathit{bound}^{\Theta, \mathbf{n}_\Theta(j)}(\psi) \\
&= \bigsqcap\{a \mid T@_{\mathbf{n}_\Theta(j)}(\psi \to a) \in \Theta\} \\
&\leq \bigsqcap\{a \mid T@_j(\psi \to a) \in \Theta\} \quad \text{(using 1 of Lemma 10)} \\
&\leq \bigsqcap\{a \mid T@_i(@_j\psi \to a) \in \Theta\} \\
&= \mathit{bound}^{\Theta, i}(@_j\psi),
\end{aligned}$$

and the @-case is done.

In case φ is $\Diamond\psi$, we need to prove that

$$\mathit{bound}_{\Theta,i}(\Diamond\psi) \leq \nu(i, \Diamond\psi) \leq \mathit{bound}^{\Theta,i}(\Diamond\psi),$$

which by definition amounts to showing that

$$\bigsqcup\{a \mid T@_i(a \to \Diamond\psi) \in \Theta\} \leq \bigsqcup\{R_\Theta(i,j) \sqcap \nu(j,\psi) \mid j \in \Theta\} \leq \bigsqcap\{a \mid T@_i(\Diamond\psi \to a) \in \Theta\}.$$

Proving the first inequality amounts to showing that if $T@_i(a \to \Diamond\psi) \in \Theta$ then

$$a \leq \bigsqcup\{R_\Theta(i,j) \sqcap \nu(j,\psi) \mid j \in \Theta\}.$$

To prove this assume toward a contradiction that

$$T@_i(a \to \Diamond\psi) \in \Theta \text{ and } a \not\leq \bigsqcup\{R_\Theta(i,j) \sqcap \nu(j,\psi) \mid j \in \Theta\},$$

for an $a \in \mathcal{H}$. Then choose a $b \in \mathcal{H}$ such that $b \geq \bigsqcup\{R_\Theta(i,j) \sqcap \nu(j,\psi) \mid j \in \Theta\}$ and b is a maximal member of $\mathcal{H}$ with $a \not\leq b$. Then by the reversal rule **(T$\geq$)**, $F@_i(\Diamond\psi \to b) \in \Theta$. Then using the **(F$\Diamond$)** rule there is a $c \in \mathcal{H}$ and a $j \in \Theta$ such that $T@_i(c \leftrightarrow \Diamond j) \in \Theta$ and $F@_j(\varphi \to (c \Rightarrow b)) \in \Theta$. Since $T@_i(c \leftrightarrow \Diamond j) \in \Theta$, Lemma 14 implies $R_\Theta(i, \mathbf{n}_\Theta(j)) = c$. Applying 1 of Lemma 10 to the formula $F@_j(\varphi \to (c \Rightarrow b)) \in \Theta$ we get $F@_{\mathbf{n}_\Theta(j)}(\varphi \to (c \Rightarrow b)) \in \Theta$. Now (iv) of Lemma 13 implies $\mathit{bound}_{\Theta,\mathbf{n}_\Theta(j)}(\psi) \not\leq c \Rightarrow b$. This further implies that $(\mathit{bound}_{\Theta,\mathbf{n}_\Theta(j)}(\psi) \sqcap c) \not\leq b$. But by the induction hypothesis $\mathit{bound}_{\Theta,\mathbf{n}_\Theta(j)}(\psi) \leq \nu(\mathbf{n}_\Theta(j), \psi)$ and thus

$$\begin{aligned}
\mathit{bound}_{\Theta,\mathbf{n}_\Theta(j)}(\psi) \sqcap c &= \mathit{bound}_{\Theta,\mathbf{n}_\Theta(j)}(\psi) \sqcap R_\Theta(i, \mathbf{n}_\Theta(j)) \\
&\leq \nu(\mathbf{n}_\Theta(j), \psi) \sqcap R_\Theta(i, \mathbf{n}_\Theta(j)) \\
&\leq \bigsqcup\{R_\Theta(i, \mathbf{n}_\Theta(j)) \sqcap \nu(\mathbf{n}_\Theta(j), \psi) \mid j \in \Theta\} \\
&\leq \bigsqcup\{R_\Theta(i,j) \sqcap \nu(j,\psi) \mid j \in \Theta\} \leq b,
\end{aligned}$$

which of course is a contradiction.

In order to prove that

$$\bigsqcup\{R_\Theta(i,j) \sqcap \nu(j,\psi) \mid j \in \Theta\} \leq \bigsqcap\{a \mid T@_i(\Diamond\psi \to a) \in \Theta\},$$

we must show that if $T@_i(\Diamond\psi \to a) \in \Theta$, then $R_\Theta(i,j) \sqcap \nu(j,\psi) \leq a$ for all $j \in \Theta$. Thus assume that $T@_i(\Diamond\psi \to a) \in \Theta$ and that $R_\Theta(i,j) \neq \bot$ (or else it's trivial) for an arbitrary $j \in \Theta$. Since $R_\Theta(i,j) \neq \bot$, the definition of R implies that j must be an urfather. Furthermore,

$$R_\Theta(i,j) = \bigsqcup\{b \mid T@_i(b \to \Diamond k) \in \Theta, \mathbf{n}_\Theta(k) = j\}.$$

Let b and k be chosen arbitrarily such that $T@_i(b \to \Diamond k) \in \Theta$ and $\mathbf{n}_\Theta(k) = j$. Then by the $(\mathbf{T}\Diamond)$ rule, $T@_k(\psi \to (b \Rightarrow a)) \in \Theta$. Using 1 of Lemma 10 we get $T@_{\mathbf{n}_\Theta(k)}(\psi \to (b \Rightarrow a)) \in \Theta$, that is, $T@_j(\psi \to (b \Rightarrow a)) \in \Theta$. Now, by induction hypothesis, since j is an urfather,

$$\nu(j,\psi) \leq \mathit{bound}^{\Theta,j}(\psi) \leq b \Rightarrow a.$$

Since k and b were chosen arbitrarily with $T@_i(b \to \Diamond k) \in \Theta$ and $\mathbf{n}_\Theta(k) = j$, we get

$$\nu(j,\psi) \leq \bigsqcap\{b \Rightarrow a \mid T@_i(b \to \Diamond k) \in \Theta, \mathbf{n}_\Theta(k) = j\}.$$

We now get

$$\begin{aligned}
R_\Theta(i,j) \sqcap \nu(j,\psi) &\leq \bigsqcup\{b \mid T@_i(b \to \Diamond k) \in \Theta, \mathbf{n}_\Theta(k) = j\} \\
&\quad \sqcap \bigsqcap\{b \Rightarrow a \mid T@_i(b \to \Diamond k) \in \Theta, \mathbf{n}_\Theta(k) = j\} \\
&\leq \bigsqcup\{b \sqcap (b \Rightarrow a) \mid T@_i(b \to \Diamond k) \in \Theta, \mathbf{n}_\Theta(k) = j\} \\
&\leq \bigsqcup\{a \mid T@_i(b \to \Diamond k) \in \Theta, \mathbf{n}_\Theta(k) = j\} \\
&\leq a.
\end{aligned}$$

Because $j \in \Theta$ was arbitrary it follows that it holds for all $j \in \Theta$ and the proof of this case is completed.

In case φ is $\Box\psi$, we need to prove that

$$\bigsqcup\{a \mid T@_i(a \to \Box\psi) \in \Theta\} \leq \bigsqcap\{R_\Theta(i,j) \Rightarrow \nu(j,\psi) \mid j \in \Theta\} \leq \bigsqcap\{a \mid T@_i(\Box\psi \to a) \in \Theta\}.$$

To prove the first inequality we need to prove that if $j \in \Theta$, then

$$a \leq R_\Theta(i,j) \Rightarrow \nu(j,\psi), \tag{2.2}$$

for all $a \in \mathcal{H}$ with $T@_i(a \to \Box\psi) \in \Theta$. So let $a \in \mathcal{H}$ be given arbitrarily such that $T@_i(a \to \Box\psi) \in \Theta$. Note that (2.2) is equivalent to

$$a \sqcap R_\Theta(i,j) \leq \nu(j,\psi).$$

By definition of R_Θ we have

$$R_\Theta(i,j) = \bigsqcup\{b \mid T@_i(b \to \Diamond k) \in \Theta, \mathbf{n}_\Theta(k) = j\}.$$

Let b and k be chosen arbitrarily such that $T@_i(b \to \Diamond k) \in \Theta$ and $\mathbf{n}_\Theta(k) = j$. Then by the **(T□)**-rule it follows that $T@_k((a \sqcap b) \to \psi) \in \Theta$. By 1 of Lemma 10 this implies $T@_j((a \sqcap b) \to \psi) \in \Theta$. Thus we get $\mathit{bound}_{\Theta,j}(\psi) \geq (a \sqcap b)$. Since b and k were chosen arbitrarily with the properties $T@_i(b \to \Diamond k) \in \Theta$ and $\mathbf{n}_\Theta(k) = j$ we then get

$$\mathit{bound}_{\Theta,j}(\psi) \geq \bigsqcup\{a \sqcap b \mid T@_i(b \to \Diamond k) \in \Theta, \mathbf{n}_\Theta(k) = j\}.$$

Using this inequality and the induction hypothesis we now get

$$\begin{aligned} a \sqcap R_\Theta(i,j) &= a \sqcap \bigsqcup\{b \mid T@_i(b \to \Diamond k) \in \Theta, \mathbf{n}_\Theta(k) = j\} \\ &= \bigsqcup\{a \sqcap b \mid T@_i(b \to \Diamond k) \in \Theta, \mathbf{n}_\Theta(k) = j\} \\ &\leq \mathit{bound}_{\Theta,j}(\psi) \leq \nu(j,\psi). \end{aligned}$$

Since a was arbitrary this holds for all $a \in \mathcal{H}$ and the inequality have been proven.

To show the other inequality we need to show that

$$\text{if } \; T@_i(\Box\psi \to a) \in \Theta \; \text{ then } \; \bigsqcap\{R_\Theta(i,j) \Rightarrow \nu(j,\psi) \mid j \in \Theta\} \leq a.$$

If $a = \top$ then this is trivial. Thus assume towards a contradiction that there is an $a \neq \top$ with $T@_i(\Box\psi \to a) \in \Theta$ and $\bigsqcap\{R_\Theta(i,j) \Rightarrow \nu(j,\psi) \mid j \in \Theta\} \not\leq a$. Now let $b \leq \bigsqcap\{R_\Theta(i,j) \Rightarrow \nu(j,\psi) \mid j \in \Theta\}$ be a minimal member of $\mathcal{H}$ such that $b \not\leq a$. Then by the reversal rule **(T≤)**, $F@_i(b \to \Box\psi) \in \Theta$. Hence by the **(F□)**-rule there is a nominal $k \in \Theta$ and a $c \in \mathcal{H}$ such that $T@_i(c \leftrightarrow \Diamond k) \in \Theta$ and $F@_k((b \sqcap c) \to \psi) \in \Theta$. From the first it follows that $R_\Theta(i, \mathbf{n}_\Theta(k)) = c$, using Lemma 14. From the second it follows that $F@_{\mathbf{n}_\Theta(k)}((b \sqcap c) \to \psi) \in \Theta$, using 1 of Lemma 10, and thus, by *(iii)* of Lemma 13, $b \sqcap c \not\leq \mathit{bound}^{\Theta,\mathbf{n}_\Theta(k)}(\psi)$. But then from the induction hypothesis it follows that

$$b \sqcap c \not\leq \nu(\mathbf{n}_\Theta(k),\psi) \leq \mathit{bound}^{\Theta,\mathbf{n}_\Theta(k)}(\psi).$$

Hence

$$b \not\leq c \Rightarrow \nu(\mathbf{n}_\Theta(k), \psi) = R_\Theta(i, \mathbf{n}_\Theta(k)) \Rightarrow \nu(\mathbf{n}_\Theta(k), \psi).$$

But by the assumption on b we also have that

$$b \leq \bigsqcap\{R_\Theta(i,j) \Rightarrow \nu(j,\psi) \mid j \in \Theta\} \leq R_\Theta(i, \mathbf{n}_\Theta(k)) \Rightarrow \nu(\mathbf{n}_\Theta(k), \psi),$$

and a contradiction have been reached. This concludes the $\Box$ case and thus the entire proof of the theorem. $\Box$

Now completeness can easily be proven, in the following sense.

Theorem 16. *If there is no tableau proof of the formula φ, then there is a model $\mathcal{M} = \langle W, R, \mathbf{n}, \nu \rangle$ and a $w \in W$ such that $\nu(w, \varphi) \neq \top$.*

Proof. Assume that there is no tableau proof of the formula φ. Then there is an saturated tableau with a open branch Θ starting with the formula $F@_i(\top \to \varphi)$ for a nominal i not in φ. By item 1 of Lemma 10 it follows that also $F@_{\mathbf{n}_\Theta(i)}(\top \to \varphi) \in \Theta$.

The model $\mathcal{M}_\Theta = \langle W_\Theta, R_\Theta, \mathbf{n}_\Theta, \nu_\Theta \rangle$ can now be constructed such that ν_Θ satisfies the assumption of Theorem 15. Since $F@_{\mathbf{n}_\Theta(i)}(\top \to \varphi) \in \Theta$ it follows by Lemma 13 that $\top \not\leq \mathit{bound}^{\Theta, \mathbf{n}_\Theta(i)}(\varphi)$. But by Theorem 15, since φ is a subformula of the root formula and $\mathbf{n}_\Theta(i)$ is an urfather, we know that $\nu_\Theta(\mathbf{n}_\Theta(i), \varphi) \leq \mathit{bound}^{\Theta, \mathbf{n}_\Theta(i)}(\varphi)$ and it thus follows that $\top \not\leq \nu_\Theta(\mathbf{n}_\Theta(i), \varphi)$ and the proof is completed. $\Box$

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Chapter 3

Alternative semantics for a many-valued hybrid logic

Unpublished manuscript.

Abstract: In the paper [83] (Chapter 2) a many-valued hybrid logic was introduced as an extension of a many-valued modal logic of Fitting [59, 60, 61]. It was argued that the choice of semantics for the hybrid machinery was the most natural generalization of the standard semantics for hybrid logic. In this paper, alternative ways of defining the semantics of the hybrid machinery are discussed and compared to semantics of [83].

Keywords: Hybrid logic, many-valued modal logic, nominals, the satisfaction operator, many-valued hybrid logic semantics.

This paper is a supplement to the paper [83] on many-valued hybrid logic. In [83] particular semantics are chosen for the hybrid part of the language. Nevertheless, other alternative definitions of the semantics are possible and in the present paper we will consider such alternatives. We will look at five other possible logics, even though many more are possible. However, the discussion of these five logics will show that the semantics chosen in [83] is presumably the most natural hybrid extension of the underlying many-valued modal logic.

We will use the same syntax for the language as in [83], and we will denote the logic of that paper by **MVHL**. As in [83] we will only consider sets of truth-values that are finite Heyting algebras. Compared to **MVHL**, we will focus on the possibility of changing only two things; the way nominals are interpreted in a model, and how the semantics of the satisfaction operator $@_i$ is defined. An exception will be the logic $\mathbf{MVHL}_3$ where we will also place an extra requirement on the Heyting algebras. Note, in particular, that we will change neither the semantics of the modalities $\Box$ and $\Diamond$, nor the semantics of the global modalities A and E.

The new logics introduced in this paper will be compared to the logic of [83] on four aspects. One key feature of hybrid logic is that it allows for equational reasoning about worlds in the language. This is due to the fact that the formula $@_i j$ expresses equality between the two worlds denoted by i and j. In the many-valued setting of [83], this feature reveals itself as the fact that

$$\nu(w, @_i j) = \top \text{ iff } \mathbf{n}(i) = \mathbf{n}(j).$$

Another key feature of hybrid logic is its ability to express accessibility between worlds, in the sense that $@_i \Diamond j$ is true if and only if the world denoted by j is accessible by the world denoted by i. In the many-valued setting of [83] this is again the case since the truth value of the accessibility between the world denoted by i and the world denoted by j, $R(\mathbf{n}(i), \mathbf{n}(j))$, is equal to the truth value of the formula $@_i \Diamond j$.

The third property of standard hybrid logic is the fact that the satisfaction operator can be defined by nominals and the global modality. This is due to the fact that the truth values of the formulas $@_i \varphi$, $E(i \wedge \varphi)$, and $A(i \rightarrow \varphi)$ are equal. This is also the case for the logic of [83]. Note that the question of whether $@_i \varphi$ can be defined as $E(i \wedge \varphi)$ and $A(i \rightarrow \varphi)$, actually consists of three questions, namely the question of whether $@_i \varphi$ is equivalent to $E(i \wedge \varphi)$, whether $@_i \varphi$ is equivalent to $A(i \rightarrow \varphi)$, and, finally, whether $E(i \wedge \varphi)$ and $A(i \rightarrow \varphi)$ are equivalent. A consequence of this paper is the realization that the answers to these three questions are independent.[1]

Finally, the fourth property of the logic of [83] that makes it a natural extension of standard hybrid logic is the fact that if the truth value set is the simple Heyting algebra $\{\top, \bot\}$, then the logic collapses to standard hybrid logic.

In the following five sections we will present five different logics that are all variations of the original **MVHL**. We will end each section with a summary on the differences and similarities between the given logic and **MVHL**, based

[1]Note that we will only compare logics on whether the satisfaction operator is definable by $E(i \wedge \varphi)$ or $A(i \rightarrow \varphi)$, not whether the satisfaction operator is definable from nominals and global modalities in general. The same issue arises in the question of defining equality or accessibility between worlds. See Section 3.7 for further discussions.

on the four mentioned properties. Our findings are simplified in Figure 3.1. In the end, we will give a short conclusion and direction for further research.

3.1 MVHL$_1$

In the semantics of **MVHL** every nominal was assigned $\top$ in one world and $\bot$ in the rest. We keep the requirement that nominals are assigned $\top$ in exactly one world, but do not put any requirement on what they are assigned in other worlds (as long as it is not $\top$). This leads to a new definition of a model; however, the only thing we change is the nominal assignment $\mathbf{n}$.

Definition 17 (Alternative notion of a model)**.** *A model $\mathcal{M}$ is a tuple $\mathcal{M} = \langle W, R, \nu, \mathbf{n}\rangle$, where $\langle W, R, \nu\rangle$ is the same as for standard **MVHL**, but now $\mathbf{n} : W \times \mathsf{NOM} \to \mathcal{H}$ and for all $i \in \mathsf{NOM}$ there is a unique $w \in W$ such that $\nu(w, i) = \top$. For all $i \in \mathsf{NOM}$, the unique w such that $\nu(w, i) = \top$ will be denoted by $\bar{i}$ and referred to as the* denotation of i.

The valuation ν can now be extended to all **MVHL**-formulas almost as before. The semantic of the nominal i at the world w is given by

$$\nu(w, i) = \mathbf{n}(w, i).$$

Furthermore, the semantic of $@_i\varphi$ needs to be changed as well and we choose to define it in the following way:

$$\nu(w, @_i\varphi) = \nu(\bar{i}, \varphi), \tag{3.1}$$

where $\bar{i}$ is the denotation of i as defined in Definition 17. By this definition the truth value of the formula $@_i\varphi$ is precisely the truth value of φ at the world denoted by the nominal i. Call the logic obtained by this semantics for **MVHL**$_1$.

How does the new logic **MVHL**$_1$ looks compared to **MVHL**? First, note that $\nu(w, i)$ still gets the value $\top$ in exactly one world, and therefore nominals still denote single worlds in a sense. As a consequence, the logic can still express equality of worlds because:

$$\nu(w, @_ij) = \top \text{ iff } \bar{i} = \bar{j},$$

for all $i, j \in \mathsf{NOM}$, all **MVHL**$_1$-models $\mathcal{M} = \langle W, R, \nu, \mathbf{n}\rangle$ and all $w \in W$. That this is true is not hard to see from the definition of the semantics.

However, the two logics do differ considerably. The accessibility between worlds are no longer expressible in the same way. Consider the following example:

Example 18. *Let $\mathcal{H} = \{\bot, a, b, \top\}$ be such that neither $a \leq b$ nor $b \leq a$, and thus $a \sqcup b = \top$.*[2] *Let the model $\mathcal{M} = \langle W, R, \nu, \mathbf{n}\rangle$ satisfy the following:*

$W = \{w_0, w_1\}$;

$R(w_0, w_0) = b,\ R(w_0, w_1) = a$;

$\mathbf{n}(w_0, i) = \top,\ \mathbf{n}(w_1, i) = \bot$;

$\mathbf{n}(w_0, j) = b,\ \mathbf{n}(w_1, j) = \top$,

for distinct nominals i and j. Then, we have for all $w \in W$ that

$$\begin{array}{rcl} \nu(w, @_i\Diamond j) & = & \nu(\bar{i}, \Diamond j) \\ & = & \bigsqcup\{R(\bar{i}, v) \sqcap \nu(v, j) \mid v \in W\} \\ & = & \big(R(\bar{i}, w_0) \sqcap \mathbf{n}(w_0, j)\big) \sqcup \big(R(\bar{i}, w_1) \sqcap \mathbf{n}(w_1, j)\big) \\ & = & b \sqcup a \\ & = & \top \\ & \neq & a \ = \ R(\bar{i}, \bar{j}). \end{array}$$

Note, however, that we do have that $R(\bar{i}, \bar{j}) \leq \nu(w, @_i\Diamond j)$ always holds.[3]

Furthermore, the semantic of $@_i\varphi$ can no longer be defined in terms of the global modalities E or A. This can be seen by the following example:

Example 19. *Let $\mathcal{H} = \{\bot, a, \top\}$ be a Heyting algebra and define the model $\mathcal{M} = \langle W, R, \nu, \mathbf{n}\rangle$, such that:*

$W = \{w_0, w_1\}$;

$\mathbf{n}(w_0, i) = \top,\ \mathbf{n}(w_1, i) = a$;

$\nu(w_0, p) = \bot, \nu(w_1, p) = a$,

[2] Note that $\mathcal{H}$ actually is a Heyting algebra, which is easy to see.

[3] In $\bigsqcup\{R(\bar{i}, v) \sqcap \nu(v, j) \mid v \in W\}$, if $v = \bar{j}$ then $R(\bar{i}, v) \sqcap \nu(v, j) = R(\bar{i}, \bar{j})$ and thus

$$R(\bar{i}, \bar{j}) \leq \bigsqcup\{R(\bar{i}, v) \sqcap \nu(v, j) \mid v \in W\} = \nu(w, @_i\Diamond j).$$

for an $i \in \mathsf{NOM}$ *and* $p \in \mathsf{PROP}$. *In this model* $\nu(w, @_i p) = \nu(\bar{i}, p) = \nu(w_0, p) = \bot$. *Nevertheless,*

$$\begin{array}{rcl} \nu(w, E(i \wedge p)) & = & \bigsqcup\{\nu(v, i) \sqcap \nu(v, p) \mid v \in W\} \\ & = & \big(\mathbf{n}(w_0, i) \sqcap \nu(w_0, p)\big) \sqcup \big(\mathbf{n}(w_1, i) \sqcap \nu(w_1, p)\big) \\ & = & \bot \sqcup a \\ & = & a, \end{array}$$

and since $\bot \neq a$, $@_i p$ *and* $E(i \wedge p)$ *are not equivalent. Actually, in this model*

$$\begin{array}{rcl} \nu(w, A(i \to p)) & = & \bigsqcap\{\nu(v, i) \Rightarrow \nu(v, p) \mid v \in W\} \\ & = & (\mathbf{n}(w_0, i) \Rightarrow \nu(w_0, p)) \sqcap (\mathbf{n}(w_1, i) \Rightarrow \nu(w_1, p)) \\ & = & \bot \sqcap \top \\ & = & \bot. \end{array}$$

However, if the values of $\nu(w_0, p)$ *and* $\nu(w_1, p)$ *are interchanged then* $\nu(w, A(i \to p)) = \bot$ *still holds, but now* $\nu(w, @_i p) = a$ *will be the case.*

The above example also shows that $E(i \wedge \varphi)$ and $A(i \to \varphi)$ are not equivalent in general in **MVHL**$_1$, as is the case for **MVHL** (in **MVHL** they are both equivalent to $@_i\varphi$). But the formulas $E(i \wedge \varphi)$ and $A(i \to \varphi)$ are still related to the formula $@_i\varphi$ since we have the following inequality in **MVHL**$_1$:

$$\nu(w, A(i \to \varphi) \leq \nu(w, @_i\varphi) \leq \nu(w, E(i \wedge \varphi)),$$

for all $w \in W$. This follows from the following calculations:

$$\begin{array}{rcl} \nu(w, A(i \to \varphi)) & = & \bigsqcap\{\nu(v, i) \Rightarrow \nu(v, \varphi) \mid v \in W\} \\ & \leq & \nu(\bar{i}, i) \Rightarrow \nu(\bar{i}, \varphi) \\ & = & \nu(\bar{i}, \varphi) \quad (= \nu(w, @_i\varphi)) \\ & = & \nu(\bar{i}, i) \sqcap \nu(\bar{i}, \varphi) \\ & \leq & \bigsqcup\{\nu(v, i) \sqcap \nu(v, \varphi) \mid v \in W\} \\ & = & \nu(w, E(i \wedge \varphi)). \end{array} \tag{3.2}$$

Observe, that if $\mathcal{H} = \{\top, \bot\}$ (i.e. we are in the classical two-valued setting), then the logic **MVHL**$_1$ reduces to the logic **MVHL**. Thus, **MVHL**$_1$ also extends classical two-valued hybrid logic.

Summary 20 (**MVHL**$_1$ vs. **MVHL**)**.** *We summarize the semantic differences and similarities between* ***MVHL***$_1$ *and* ***MVHL***:

- *As was the case for* $\boldsymbol{MVHL}$*, equality between worlds can be expressed in* $\boldsymbol{MVHL}_1$ *since the following holds in* $\boldsymbol{MVHL}_1$*:*

$$\nu(w, @_i j) = \top \quad \textit{iff} \quad \bar{i} = \bar{j}$$

- *However, accessibility between worlds is not expressible in* $\boldsymbol{MVHL}_1$*, in the sense that* $\nu(w, @_i \Diamond j) = R(\bar{i}, \bar{j})$ *does not hold in general. We do have* $R(\bar{i}, \bar{j}) \leq \nu(w, @_i \Diamond j)$ *though.*

- *Furthermore,* $@_i\varphi$ *is not definable from the global modalities in the usual way. Still, we have the following*

$$\nu(w, A(i \to \varphi) \leq \nu(w, @_i\varphi) \leq \nu(w, E(i \wedge \varphi)).$$

  None of the inequalities can be replaced by equality in general. As a byproduct, $A(i \to \varphi)$ *and* $E(i \wedge \varphi)$ *are not equivalent in* $\boldsymbol{MVHL}_1$*.*

- *Finally, as for* $\boldsymbol{MVHL}$*,* $\boldsymbol{MVHL}_1$ *collapses to standard two-valued hybrid logic in the case* $\mathcal{T} = \{\top, \bot\}$*.*

3.2 $\mathbf{MVHL}_2$

In $\mathbf{MVHL}_1$ the definability of $@_i\varphi$ in terms of the global modalities fails. However, if we want to take this problem seriously, we could just change the semantic of $@_i\varphi$ such that it matches that of $E(i \wedge \varphi)$.[4] This we will now do. Call the logic obtained by replacing the semantic definition (3.1) by (3.3) $\mathbf{MVHL}_2$, where

$$\nu(w, @_i\varphi) = \bigsqcup \{\nu(v, i) \sqcap \nu(v, \varphi) \mid v \in W\}. \tag{3.3}$$

Then, it follows directly from the definition that in $\mathbf{MVHL}_2$, $\nu(w, @_i\varphi) = \nu(w, E(i \wedge \varphi))$.[5] Now, what about the other properties of this logic? This time it is the characterization of equality of worlds by $@_i j$ that fails. Note, that the notion of a model for $\mathbf{MVHL}_2$ is the same as for $\mathbf{MVHL}_1$, i.e. Definition 17. Now, consider the following example:

Example 21. *Let* $\mathcal{H}$ *be as in Example 18 and let the model* $\mathcal{M} = \langle W, R, \nu, \mathbf{n}\rangle$ *satisfy:*

[4] Alternatively, we could have changed the semantic of $@_i\varphi$ such that it matches that of $A(i \to \varphi)$. This we will do in Section 3.4.

[5] As before, we still have that $\nu(w, A(i \to \varphi)) \leq \nu(w, @_i\varphi)$ and that equality does not hold in general.

$W = \{w_0, w_1, w_2, w_3\}$;

$\mathbf{n}(w_0, i) = a$, $\mathbf{n}(w_1, i) = b$, $\mathbf{n}(w_2, i) = \bot$, $\mathbf{n}(w_3, i) = \top$;

$\mathbf{n}(w_0, j) = a$, $\mathbf{n}(w_1, j) = b$, $\mathbf{n}(w_2, j) = \top$, $\mathbf{n}(w_3, j) = \bot$,

for distinct nominals i and j. In this model we have for all $w \in W$ that

$$\begin{aligned}
\nu(w, @_i j) &= \bigsqcup\{\nu(v, i) \sqcap \nu(v, j) \mid v \in W\} \\
&= \big(\mathbf{n}(w_0, i) \sqcap \mathbf{n}(w_0, j)\big) \sqcup \big(\mathbf{n}(w_1, i) \sqcap \mathbf{n}(w_1, j)\big) \\
&\quad \sqcup \big(\mathbf{n}(w_2, i) \sqcap \mathbf{n}(w_2, j)\big) \sqcup \big(\mathbf{n}(w_3, i) \sqcap \mathbf{n}(w_3, j)\big) \\
&= a \sqcup b \sqcup \bot \sqcup \bot = \top,
\end{aligned}$$

however, $\bar{i} = w_3 \neq w_2 = \bar{j}$.

$\nu(w, @_i j) = \top$ does not guarantee that i and j denote the same world, nevertheless, the other implication, "if $\bar{i} = \bar{j}$ then $\nu(w, @_i j) = \top$", does hold.

Now, take the model of Example 18 again. With the **MVHL**$_2$ semantics we have that

$$\begin{aligned}
\nu(w, @_i \Diamond j) &= \bigsqcup\{\nu(v, i) \sqcap \nu(v, \Diamond j) \mid v \in W\} \\
&= \big(\mathbf{n}(w_0, i) \sqcap \nu(w_0, \Diamond j)\big) \sqcup \big(\mathbf{n}(w_1, i) \sqcap \nu(w_1, \Diamond j)\big) \\
&= \Big(\top \sqcap \bigsqcup\{R(w_0, u) \sqcap \nu(u, j) \mid u \in W\}\Big) \\
&\quad \sqcup \Big(\bot \sqcap \bigsqcup\{R(w_1, u) \sqcap \nu(u, j) \mid u \in W\}\Big) \\
&= \Big(\big(R(w_0, w_0) \sqcap \mathbf{n}(w_0, j)\big) \sqcup \big(R(w_0, w_1) \sqcap \mathbf{n}(w_1, j)\big)\Big) \sqcup \bot \\
&= b \sqcup a \\
&= \top,
\end{aligned}$$

however, $R(\bar{i}, \bar{j}) = a \neq \top$. This shows that the formula $@_i \Diamond j$ does not express accessibility between the worlds denoted by $\bar{i}$ and $\bar{j}$ in **MVHL**$_2$. Nonetheless, we still have the inequality $R(\bar{i}, \bar{j}) \leq \nu(w, @_i \Diamond j)$.[6]

Finally, note that if $\mathcal{H} = \{\top, \bot\}$, then

$$\nu(w, @_i \varphi) = \bigsqcup\{\nu(v, i) \sqcap \nu(v, \varphi) \mid v \in W\} = \nu(\bar{i}, i) \sqcap \nu(\bar{i}, \varphi) = \nu(\bar{i}, \varphi),$$

Thus, in the case $\mathcal{H} = \{\top, \bot\}$, **MVHL**$_2$ collapses to **MVHL**$_1$ and thus also to standard two-valued hybrid logic.

[6] This is because: $\nu(w, @_i \Diamond j) = \bigsqcup\{\nu(v, i) \sqcap \nu(v, \Diamond j) \mid v \in W\} = \bigsqcup\{\mathbf{n}(v, i) \sqcap \big(\bigsqcup\{R(v, u) \sqcap \mathbf{n}(u, j) \mid u \in W\}\big) \mid v \in W\} = \bigsqcup\{\bigsqcup\{\mathbf{n}(v, i) \sqcap R(v, u) \sqcap \mathbf{n}(u, j) \mid u \in W\} \mid v \in W\} \geq \mathbf{n}(\bar{i}, i) \sqcap R(\bar{i}, \bar{j}) \sqcap \mathbf{n}(\bar{j}, j) = R(\bar{i}, \bar{j})$.

Summary 22 (**MVHL**$_2$ vs. **MVHL**)**.** *We summarize the semantic differences and similarities between* $\boldsymbol{MVHL}_2$ *and* $\boldsymbol{MVHL}$*:*

- *Contrary to* $\boldsymbol{MVHL}$*, equality between worlds cannot be expressed by the formula* $@_i j$ *in* $\boldsymbol{MVHL}_2$*.*
- *Accessibility between worlds is not expressible in* $\boldsymbol{MVHL}_2$ *either, since* $\nu(w, @_i\Diamond j) = R(\bar{i}, \bar{j})$ *does not hold in general, though* $R(\bar{i}, \bar{j}) \leq \nu(w, @_i\Diamond j)$ *still holds.*
- *We have that* $@_i\varphi$ *is equivalent to* $E(i\wedge\varphi)$ *by definition, but the inequality*

$$\nu(w, A(i \to \varphi)) \leq \nu(w, E(i \wedge \varphi))$$

  cannot be replaced by equality in general for $\boldsymbol{MVHL}_2$ *either.*
- *Finally, as for* $\boldsymbol{MVHL}$*,* $\boldsymbol{MVHL}_2$ *collapses to standard two-valued hybrid logic in the case* $\mathcal{T} = \{\top, \bot\}$*.*

3.3 MVHL$_3$

It turns out that the lack of power in **MVHL**$_2$ to express equality of worlds using nominals can be fixed by placing a requirement on the Heyting algebra $\mathcal{H}$. First a definition (adopted from [48], page 53):

Definition 23. *Let* $\mathcal{H}$ *be a finite Heyting algebra. An element* $a \in \mathcal{H}$ *is called* join-irreducible *if* $a \neq \bot$ *and for all* $x, y \in \mathcal{H}$*,* $a = x \sqcup y$ *implies that* $a = x$ *or* $a = y$*.*

The requirement that will be placed on the Heyting algebra $\mathcal{H}$ is that $\top$ is join-irreducible. Note that, if $\mathcal{H} = \{\top, \bot\}$, then $\top$ is join-irreducible. Moreover, if $\mathcal{H}$ is a linear ordered finite Heyting algebra, then $\top$ is also join-irreducible. Thus, in the Heyting algebra of Example 19 $\top$ is join-irreducible whereas $\top$ is not join-irreducible in Heyting algebra of Example 18. We now have the following result:

Theorem 24. *For all* $\boldsymbol{MVHL}_2$ *models* $\mathcal{M} = \langle W, R, \nu, \mathbf{n}\rangle$ *build on a Heyting algebra where* $\top$ *is join-irreducible, the following holds:*

$$\nu(w, @_i j) = \top \quad \textit{iff} \quad \bar{i} = \bar{j}, \tag{3.4}$$

for all $i, j \in \mathsf{NOM}$ *and all* $w \in W$*.*

Proof. Assume that $\mathcal{M} = \langle W, R, \nu, \mathbf{n}\rangle$ is a model built on a Heyting algebra where $\top$ is join-irreducible. As mentioned before the "if" part is easy and always holds. So assume now that $\nu(w, @_i j) = \top$ for a $w \in W$ and two nominals i and j. Then we need to prove that $\bar{i} = \bar{j}$. However, since

$$\top = \nu(w, @_i j) = \bigsqcup \{\nu(v, i) \sqcap \nu(v, j) \mid v \in W\},$$

and $\top$ is join-irreducible, there must be a $v \in W$ such that $\nu(v,i) \sqcap \nu(v,j) = \top$. This again is only possible if both $\nu(v, i) = \top$ and $\nu(v, j) = \top$, and thus $\bar{i} = \bar{j}$. $\square$

There is a sense in which the other "implication" of Theorem 24 is also true: We can characterize the finite Heyting algebras where $\top$ is join-irreducible by the property (3.4). Thus we have a characterization not of a class of frames but a class of Heyting algebras acting as premissible sets of truth values:

Theorem 25. *For a finite Heyting algebra $\mathcal{H}$, $\top$ is join-irreducible if and only if the following property holds:*

> $(*)$ *For all $\boldsymbol{MVHL}_2$-models $\mathcal{M} = \langle W, R, \nu, \mathbf{n}\rangle$, for all $w \in W$, and for all nominals i and j*
> $$\nu(w, @_i j) = \top \quad \textit{iff} \quad \bar{i} = \bar{j}.$$

Proof. Then one direction, namely the fact that "if $\top$ is join-irreducible in $\mathcal{H}$ then $(*)$ holds" is a direct consequence of Theorem 24. For the other direction assume that $\mathcal{H}$ is a finite Heyting algebra satisfying $(*)$. Assume towards a contradiction that $\top$ is not join-irreducible in $\mathcal{H}$. Then there must be two distinct elements a and b of $\mathcal{H}$ strictly below $\top$ such that neither $a \leq b$ or $b \leq a$, but $a \sqcup b = \top$.

Now define a **MVHL**$_2$ model $\mathcal{M} = \langle W, R, \nu, \mathbf{n}\rangle$ such that

$$W = \{w_0, w_1\};$$

$$\mathbf{n}(w_0, i) = \top, \ \mathbf{n}(w_1, i) = a;$$

$$\mathbf{n}(w_0, j) = b, \ \mathbf{n}(w_1, j) = \top,$$

for two distinct nominals i and j. Then

$$\begin{aligned}
\nu(w, @_i j) &= \bigsqcup \{\nu(v, i) \sqcap \nu(v, j) \mid v \in W\} \\
&= \big(\mathbf{n}(w_0, i) \sqcap \mathbf{n}(w_0, j)\big) \sqcup \big(\mathbf{n}(w_1, i) \sqcap \mathbf{n}(w_1, j)\big) \\
&= (\top \sqcap b) \sqcup (a \sqcap \top) \\
&= b \sqcup a = \top,
\end{aligned}$$

and since $\mathcal{H}$ satisfies $(*)$ it follows that $\bar{i} = \bar{j}$. However, this is a contradiction since in $\mathcal{M}$, $\bar{i} = w_0 \neq w_1 = \bar{j}$. Thus, $\top$ must be join-irreducible in $\mathcal{H}$ and the proof is completed. $\square$

The logic obtained from $\mathbf{MVHL}_2$ by only allowing finite Heyting algebras where $\top$ is join-irreducible will be denoted by $\mathbf{MVHL}_3$. Again, the notion of a $\mathbf{MVHL}_3$-model is the same as in Definition 17.

We now look at the formula $@_i\Diamond j$ and the possibility of defining accessibility between worlds in $\mathbf{MVHL}_3$. Assume that $\top$ is join-irreducible in the Heyting algebra $\mathcal{H}$. Because,

$$\nu(w, @_i\Diamond j) = \bigsqcup\{\nu(v, i) \sqcap \nu(v, \Diamond j) \mid v \in W\},$$

$\nu(w, @_i\Diamond j)$ can only be $\top$ if there is a $v \in W$ such that $\nu(v, i) \sqcap \nu(v, \Diamond j) = \top$. Thus, we have for $\mathbf{MVHL}_3$ that

$$\nu(w, @_i\Diamond j) = \top \quad \textit{iff} \quad R(\bar{i}, \bar{j}) = \top.$$

This is contrary to $\mathbf{MVHL}_2$ where we showed that in Example 18, $R(\bar{i}, \bar{j}) = a$, but $\nu(w, @_i\Diamond j) = \top$. Furthermore, as for $\mathbf{MVHL}_1$ and $\mathbf{MVHL}_2$ we have that the inequality $R(\bar{i}, \bar{j}) \leq \nu(w, @_i\Diamond j)$ holds in general. Nevertheless, if $R(\bar{i}, \bar{j}) < \top$, then $R(\bar{i}, \bar{j}) < \nu(w, @_i\Diamond j)$ can occur in $\mathbf{MVHL}_3$, as the following example shows:

Example 26. *Let $\mathcal{H} = \{\top, a, b, \bot\}$ be a Heyting algebra such that $\bot < b < a < \top$. Note that, since $\mathcal{H}$ is a linear order, $\top$ is join-irreducible. Let the model $\mathcal{M} = \langle W, R, \nu, \mathbf{n}\rangle$ satisfy:*

$W = \{w_0, w_1\}$;

$R(w_0, w_0) = a,\ R(w_0, w_1) = b$;

$\mathbf{n}(w_0, i) = \top,\ \mathbf{n}(w_1, i) = a$;

$\mathbf{n}(w_0, j) = a,\ \mathbf{n}(w_1, j) = \top$,

for distinct nominals i and j. In this model we have

$$\begin{aligned}
\nu(w, @_i\Diamond j) &= \bigsqcup\{\nu(v, i) \sqcap \nu(v, \Diamond j) \mid v \in W\} \\
&= \bigsqcup\{\mathbf{n}(v, i) \sqcap \big(\bigsqcup\{R(v, u) \sqcap \mathbf{n}(u, j) \mid u \in W\}\big) \mid v \in W\} \\
&= \bigsqcup\{\bigsqcup\{\mathbf{n}(v, i) \sqcap R(v, u) \sqcap \mathbf{n}(u, j) \mid u \in W\} \mid v \in W\} \\
&\geq \mathbf{n}(w_0, i) \sqcap R(w_0, w_0) \sqcap \mathbf{n}(w_0, j) \\
&= \top \sqcap a \sqcap a = a,
\end{aligned}$$

and thus, $R(\bar{i},\bar{j}) = b < a \leq \nu(w, @_i\Diamond j)$.

At last, note that since nothing has been changed in the semantics of $@_i$ and E relative to **MVHL**$_2$, the formulas $@_i\varphi$ and $E(i\wedge\varphi)$ are still equivalent. The equality $\nu(w, A(i \to \varphi)) \leq \nu(w, E(i \wedge \varphi))$ also holds for **MVHL**$_3$ and Example 19 can be used again to show that this inequality can be strict.

Summary 27 (**MVHL**$_3$ vs. **MVHL**)**.** *We summarize the semantic differences and similarities between* ***MVHL***$_3$ *and* ***MVHL****:*

- *As in* ***MVHL*** *(contrary to* ***MVHL***$_2$*), equality between worlds can be expressed by the formula* $@_ij$ *in* ***MVHL***$_3$*.*
- *Accessibility between worlds is not generally expressible in* ***MVHL***$_3$ *either, since* $\nu(w, @_i\Diamond j) = R(\bar{i},\bar{j})$ *does not hold in general. However, in* ***MVHL***$_3$ *we have that* $R(\bar{i},\bar{j}) \leq \nu(w, @_i\Diamond j)$ *and furthermore that*
$$\nu(w, @_i\Diamond j) = \top \quad \textit{iff} \quad R(\bar{i},\bar{j}) = \top.$$
- *We have that* $@_i\varphi$ *is equivalent to* $E(i\wedge\varphi)$ *by definition, but the inequality*
$$\nu(w, A(i \to \varphi)) \leq \nu(w, E(i \wedge \varphi))$$
cannot be replaced by equality in general for ***MVHL***$_3$ *either.*
- *Finally, as for* ***MVHL***, ***MVHL***$_3$ *collapses to standard two-valued hybrid logic in the case* $\mathcal{T} = \{\top, \bot\}$*.*

3.4 MVHL$_4$

In **MVHL**$_2$ we chose to make the semantic of $@_i\varphi$ equal that of $E(i\wedge\varphi)$. We now make the semantic of $@_i\varphi$ equal to that of $A(i \to \varphi)$. Thus, let the logic **MVHL**$_4$ be the logic obtained from **MVHL**$_2$ by replacing the semantic of $@_i\varphi$ (given in (3.3)), with the following definition:

$$\nu(w, @_i\varphi) = \bigsqcap\{\nu(v, i) \Rightarrow \nu(v, \varphi) \mid v \in W\}. \tag{3.5}$$

At first glance, one may except that a logic similar to **MVHL**$_2$ will be the result. However, there are some interesting differences. In fact, equality between worlds becomes expressible now. Let a model $\mathcal{M} = \langle W, R, \nu, \mathbf{n}\rangle$ be given. If $\nu(w, @_ij) = \top$ then by the definition (3.5) it follows that:

$$\nu(v, i) \Rightarrow \nu(v, j) = \top\ , \quad \text{for all } v \in W.$$

This further implies that:

$$\nu(v,i) \leq \nu(v,j)\ , \quad \text{for all } v \in W, \tag{3.6}$$

but if $\bar{i} \neq \bar{j}$, then there is an $a \in \mathcal{H}$ such that $\nu(\bar{i},j) = a < \top = \nu(\bar{i},i)$. Hence, if $\nu(w,@_i j) = \top$ it follows that $\bar{i} = \bar{j}$. Yet, an important thing to note is that $\bar{i} = \bar{j}$ does not imply that $\nu(w,@_i j) = \top$. What is required for $\nu(w,@_i j) = \top$ to be true, is the stronger requirement that (3.6) is satisfied. That $@_i j$ expresses an inequality instead of equality is also evident from the fact that $@_i j$ and $@_j i$ are not equivalent formulas in $\mathbf{MVHL}_4$. All of this is made clear in the following example:

Example 28. *Let $\mathcal{H}$ be as in Example 18. And let the model $\mathcal{M} = \langle W, R, \nu, \mathbf{n}\rangle$ satisfy:*

$W = \{w_0, w_1\}$;

$\mathbf{n}(w_0,i) = \top,\ \mathbf{n}(w_1,i) = a,\ \mathbf{n}(w_0,j) = \top,\ \mathbf{n}(w_1,j) = b,$

for distinct nominals i and j. Then $\bar{i} = \bar{j}$, although

$$\begin{aligned}
\nu(w,@_i j) &= (\nu(w_0,i) \Rightarrow \nu(w_0,j)) \sqcap (\nu(w_1,i) \Rightarrow \nu(w_1,j)) \\
&= (\top \Rightarrow \top) \sqcap (a \Rightarrow b) \\
&= \top \sqcap b = b \\
\nu(w,@_j i) &= (\nu(w_0,j) \Rightarrow \nu(w_0,i)) \sqcap (\nu(w_1,j) \Rightarrow \nu(w_1,i)) \\
&= (\top \Rightarrow \top) \sqcap (b \Rightarrow a) \\
&= \top \sqcap a = a
\end{aligned}$$

We now inspect the formula $@_i\Diamond j$. With the Heyting algebra and model of Example 18 we have that:

$$\begin{aligned}
\nu(w,@_i\Diamond j) &= \bigsqcap\{\nu(v,i) \Rightarrow \nu(v,\Diamond j) \mid v \in W\} \\
&= (\nu(w_0,i) \Rightarrow \nu(w_0,\Diamond j)) \sqcap (\nu(w_1,i) \Rightarrow \nu(w_1,\Diamond j)) \\
&= (\top \Rightarrow \bigsqcup\{R(w_0,u) \sqcap \nu(u,j) \mid u \in W\}) \\
&\quad \sqcap (\bot \Rightarrow \bigsqcup\{R(w_1,u) \sqcap \nu(u,j) \mid u \in W\}) \\
&= \bigsqcup\{R(w_0,u) \sqcap \nu(u,j) \mid u \in W\} \sqcap \top \\
&= (R(w_0,w_0) \sqcap \nu(w_0,j)) \sqcup (R(w_0,w_1) \sqcap \nu(w_1,j)) \\
&= (b \sqcap b) \sqcup (a \sqcap \top) \\
&= b \sqcup a = \top.
\end{aligned}$$

However, $R(\bar{i}, \bar{j}) = a$. Hence, as for **MVHL**$_2$, $@_i\Diamond j$ does not express accessibility between the worlds denoted by i and j.

Since nothing has been changed for the global modalities, when we moved from **MVHL**$_2$ to **MVHL**$_4$, the inequality

$$\nu(w, A(i \to \varphi)) \leq \nu(w, E(i \wedge \varphi))$$

still holds and it cannot be replaced by equality in general. Furthermore, it is obvious that $@_i\varphi$ is equivalent to $A(i \to \varphi)$ by definition.

At last, it is not hard to see that also **MVHL**$_4$ collapses to the standard two-valued hybrid logic in the case when $\mathcal{H} = \{\top, \bot\}$. Once again we summarize the logic:

Summary 29 (**MVHL**$_4$ vs. **MVHL**)**.** *We summarize the semantic differences and similarities between* ***MVHL***$_4$ *and* ***MVHL****:*

- *In* ***MVHL***$_4$*, equality between worlds can be expressed in some sense since* $\nu(w, @_i j) = \top$ *implies that* $\bar{i} = \bar{j}$*, but* $\bar{i} = \bar{j}$ *does not implies that* $\nu(w, @_i j) = \top$ *in general. In addition,* $@_i j$ *and* $@_j i$ *are not even equivalent.*

- *Accessibility between worlds is not generally expressible in* ***MVHL***$_4$*, since* $\nu(w, @_i\Diamond j) = R(\bar{i}, \bar{j})$ *is not the case in general.*

- $@_i\varphi$ *is equivalent to* $A(i \to \varphi)$ *by definition, but the inequality*

$$\nu(w, A(i \to \varphi)) \leq \nu(w, E(i \wedge \varphi))$$

  can still not be replaced by equality in general for ***MVHL***$_4$*.*

- *Finally, as for* ***MVHL****,* ***MVHL***$_4$ *collapses to standard two-valued hybrid logic in the case* $\mathcal{T} = \{\top, \bot\}$*.*

3.5 MVHL$_5$

The different logics discussed so far have all been based on the same notion of a model, the one given in Definition 17. We will now change the notion of a model. Instead of letting nominals point out single worlds, we let them point out sets of worlds whose truth-values of the nominal "sum up" to $\top$. This is spelled out in details in the following definition:

Definition 30 (Alternative$_2$ definition of models)**.** *A model $\mathcal{M} = \langle W, R, \nu, \mathbf{n}\rangle$ is as in Definition 17, but the requirement on* $\mathbf{n}$ *has been changed to:*

> *For all nominals $i \in \mathsf{NOM}$, there is a unique finite set $W_i \subseteq W$ such that*
>
> *i)* $\bigsqcup_{w \in W_i} \mathbf{n}(w, i) = \top$.
>
> *ii) There is no proper subset of W_i for which i) holds.*
>
> *iii) For all $w \notin W_i$, $\mathbf{n}(w, i) = \bot$.*

Note, that models for the original logic **MVHL** can also be viewed as models according to this definition. Now, we can interpret the language of **MVHL** over this new class of models. To see which kind of logic that results from this, the problem of defining the semantic for $@_i\varphi$ needs to be attended. We choose the following semantic for $@_i\varphi$:

$$\nu(w, @_i\varphi) = \bigsqcup\{\nu(v, i) \sqcap \nu(v, \varphi) \mid v \in W_i\}, \tag{3.7}$$

where W_i is the finite set of Definition 30. We will define **MVHL$_5$** to be the logic obtained by interpreting the **MVHL** language only over alternative$_2$ models where the truth definition of $@_i\varphi$ is given by (3.7). Note, that due to the requirement *iii*) of Definition 30, the definition (3.7) is equivalent to the one of **MVHL$_2$** and **MVHL$_3$** given in (3.3). Thus, it further follows that in **MVHL$_5$**, $@_i\varphi$ is equivalent to $E(i \wedge \varphi)$. Now, however, the "trick" with $\bar{i}$ we used in (3.2), to prove the inequality $\nu(w, A(i \to \varphi)) \leq \nu(w, E(i \wedge \varphi))$, cannot be used in **MVHL$_5$**. In fact, it turns out that there is no general inequality or equality between the formulas $E(i \wedge \varphi)$ and $A(i \to \varphi)$. This is shown by the following example:

Example 31. *Let $\mathcal{H} = \{\bot, a, b, \top\}$ be a Heyting algebra such that neither $a \leq b$ nor $b \leq a$, and thus $a \sqcup b = \top$, $a \sqcap b = \bot$. Now, let the model $\mathcal{M} = \langle W, R, \nu, \mathbf{n}\rangle$ satisfy the following:*

> $W = \{w_0, w_1, w_2\}$;
>
> $\mathbf{n}(w_0, i) = a,\ \mathbf{n}(w_1, i) = b,\ \mathbf{n}(w_2, i) = \bot$;
>
> $\mathbf{n}(w_0, j) = \bot,\ \mathbf{n}(w_1, j) = a,\ \mathbf{n}(w_2, j) = b$;,

for distinct nominals i and j. Given a propositional variable p, we have that

$$\begin{aligned}
\nu(w, E(i \wedge p)) &= \bigsqcup\{\nu(v, i) \sqcap \nu(v, p) \mid v \in W\} = (a \sqcap \nu(w_0, p)) \sqcup (b \sqcap \nu(w_1, p));\\
\nu(w, A(i \to p)) &= \bigsqcap\{\nu(v, i) \Rightarrow \nu(v, p) \mid v \in W\} = (a \Rightarrow \nu(w_1, p)) \sqcap (b \Rightarrow \nu(w_2, p)).
\end{aligned}$$

If we let the semantic of p *be given by* $\nu(w_0, p) = \top,\ \nu(w_1, p) = \bot,\ \nu(w_2, p) = \bot$*, then we have:*

$$\begin{aligned}\nu(w, E(i \wedge p)) &= (a \sqcap \top) \sqcup (b \sqcap \bot) = a \sqcup \bot = a;\\ \nu(w, A(i \to p)) &= (a \Rightarrow \bot) \sqcap (b \Rightarrow \bot) = b \sqcap a = \bot.\end{aligned}$$

On the other hand, if we let the semantic of p *be given by* $\nu(w_0, p) = \top,\ \nu(w_1, p) = a,\ \nu(w_2, p) = \top$*, we have:*

$$\begin{aligned}\nu(w, E(i \wedge p)) &= (a \sqcap \top) \sqcup (b \sqcap a) = a \sqcup \bot = a;\\ \nu(w, A(i \to p)) &= (a \Rightarrow a) \sqcap (b \Rightarrow \top) = \top \sqcap \top = \top.\end{aligned}$$

Since nominals no longer denote single worlds, we cannot talk about equality of worlds at all. However, we can speak about equality between sets of worlds. Let i and j be distinct nominals and let W_i and W_j be as in Definition 30. Then,

$$\begin{aligned}\nu(w, @_i j) &= \bigsqcup\{\nu(v, i) \sqcap \nu(v, j) \mid v \in W_i\}\\ &= \bigsqcup\{\nu(v, i) \sqcap \nu(v, j) \mid v \in (W_i \cap W_j)\},\end{aligned}$$

because of requirement $iii)$ of Definition 30. But then because of requirement $i)$ and $ii)$ it follows that

$$\nu(w, @_i j) = \top \quad \textit{iff} \quad W_i = W_j.$$

Hence, in **MVHL**$_5$ the formula $@_i j$ expresses the equality between the sets W_i and W_j.

Assume now that $\mathcal{H} = \{\top, \bot\}$. Then, for every nominal $i \in \mathsf{NOM}$ we have that W_i must be a singleton in every model, otherwise the model would not satisfy Definition 30. The definition of the semantic of $@_i\varphi$ also collapses to the usual one for hybrid logic and thus, once again we are dealing with a logic that collapses to the standard two-valued hybrid logic when $\mathcal{T} = \{\top, \bot\}$.

Finally, we now turn to the formula $@_i\Diamond j$. At first glance it seems like the formula $@_i\Diamond j$ expresses a form of "weighted sum" of accessibility of the W_j worlds from the W_i worlds since

$$\begin{aligned}\nu(w, @_i\Diamond j) &= \bigsqcup\{\nu(v, i) \sqcap \nu(v, \Diamond j) \mid v \in W_i\}\\ &= \bigsqcup\{\nu(v, i) \sqcap \bigsqcup\{R(v, t) \sqcap \nu(t, j) \mid t \in W\} \mid v \in W_i\}\\ &= \bigsqcup\{\bigsqcup\{\nu(v, i) \sqcap R(v, t) \sqcap \nu(t, j) \mid t \in W\} \mid v \in W_i\}\\ &= \bigsqcup\{\nu(v, i) \sqcap R(v, t) \sqcap \nu(t, j) \mid (v, t) \in W_i \times W_j\}.\end{aligned}$$

However, which notion of accessibility between sets of worlds, based on an underlying accessibility relation between worlds, is the right notion is not obvious. Thus, at the current state, we do not gain much from evaluating $\mathbf{MVHL}_5$ on the basis of the formulas $@_i\Diamond j$.

Summary 32 (**MVHL**$_5$ vs. **MVHL**)**.** *We summarize the semantic differences and similarities between* $\boldsymbol{MVHL}_5$ *and* $\boldsymbol{MVHL}$*:*

- *In* $\boldsymbol{MVHL}_5$ *we cannot express equality between worlds (contrary to* $\boldsymbol{MVHL}$*), but we can express equality between sets of worlds, which seems to be the only reasonable thing given the interpretation of nominals in a model.*

- *Since we cannot talk about worlds, we cannot express accessibility between worlds. However, we can express some sort of accessibility between sets of worlds. What this accessibility amounts to is still unclear.*

- *In* $\boldsymbol{MVHL}_5$ *the formula* $@_i\varphi$ *is equivalent to* $E(i \wedge \varphi)$ *as in* $\boldsymbol{MVHL}$*. Nevertheless, there are in general no equality or even inequality between the formulas* $\nu(w, A(i \to \varphi))$ *and* $\nu(w, E(i \wedge \varphi))$*.*

- *Finally, as for all the other versions of* $\boldsymbol{MVHL}$*,* $\boldsymbol{MVHL}_5$ *collapses to standard two-valued hybrid logic in the case* $\mathcal{T} = \{\top, \bot\}$*.*

3.6 Still more logics!

In $\mathbf{MVHL}_5$ we chose to let the semantic of $@_i\varphi$ be equal to that of $E(i \wedge \varphi)$. Yet another logic could be obtained by letting the semantic of $@_i\varphi$ be equal to that of $A(i \to \varphi)$. This we will not do in details here though. It would not change the relationship between the semantics of $E(i \wedge \varphi)$ and $A(i \to \varphi)$ and it will still be an extension of standard two valued hybrid logic. The formula $@_i\Diamond j$ will receive a new semantic, but we still do not have any concept to compare it against. Finally, what the formula $@_i j$ expresses is still a little unclear.

There are still more possible variations of $\mathbf{MVHL}_5$. One obvious way to go would be to drop requirement *iii*) of Definition 30. However, we have seen enough logics, and we will leave this for further research.

3.7 Concluding remarks and further research

In this paper, we presented five alternative many-valued hybrid logics, which were all obtained by making small changes to the many-valued hybrid logic of

	$\mathbf{MVHL}_1$	$\mathbf{MVHL}_2$	$\mathbf{MVHL}_3$	$\mathbf{MVHL}_4$	$\mathbf{MVHL}_5$
$\nu(w, @_i j) = \top$ *iff* $\bar{i} = \bar{j}$ holds in general	yes	no	yes	no ($\Rightarrow$ holds)	no / yes
$\nu(w, @_i \Diamond j) = R(\bar{i}, \bar{j})$ holds in general	no ($\geq$ holds)	no ($\geq$ holds)	no ($\geq$ holds)	no	no (undef.)
$\nu(w, A(i \to \varphi)) = \nu(w, @_i \varphi)$ holds in general	no ($\leq$ holds)	no ($\leq$ holds)	no ($\leq$ holds)	yes (by def.)	no
$\nu(w, @_i \varphi) = \nu(w, E(i \wedge \varphi))$ holds in general	no ($\leq$ holds)	yes (by def.)	yes (by def.)	no ($\leq$ holds)	yes (by def.)
Collapses to **MVHL** when $\mathcal{H} = \{\top, \bot\}$	yes	yes	yes	yes	yes

Figure 3.1: Comparison between the logics $\mathbf{MVHL}_1$-$\mathbf{MVHL}_5$

[83]. We compared all the logics to the one in [83] on four matters; how well they were capable of expressing equality between worlds, how well they were capable of expressing the accessibility between worlds, how the semantic of the formula $@_i\varphi$ related to the semantics of the formulas $E(i \wedge \varphi)$ and $A(i \to \varphi)$, and whether they could be viewed as extensions of the standard two-valued hybrid logic. Our findings are summarized in Figure 3.1.

The five new logics were all extensions of standard two-valued hybrid logic, but all of them differed from the logic of [83] with respect to at least one of the other matters. Thus, if one argues for a many-valued extension of standard two-valued hybrid logic, in which equality between worlds and accessibility between worlds are expressible and where the formulas $@_i\varphi$, $E(i \wedge \varphi)$, and $A(i \to \varphi)$ are all equivalent, the logic of [83] seems to be the only reasonable choice.

When comparing the logics we showed that in several of them the formulas $@_i\varphi$, $E(i \wedge \varphi)$, and $A(i \to \varphi)$ were not all equivalent. This, however, does not excludes that the satisfaction operator $@_i$ can be defined from the global modalities in other ways. The same goes for the problems of defining accessibility or equality between worlds in the logics – we have only discussed whether the standard way of doing it is possible. To answer the question of whether the satisfaction operator $@_i$ is at all definable by nominals and global modalities in a logic, an additional proof is needed. An obvious way of proving this would be through a notion of *bisimulation* for the logic. Such definitions we leave for further research though.

Whether the logic of [83] is the only natural many-valued extension of hybrid logic, is still not entirely clear. Letting nominals denote single worlds, as done in the logics **MVHL** and $\mathbf{MVHL}_1$-$\mathbf{MVHL}_4$, may not be the most natural thing in a many-valued setting. In some sense all these logics treat the nominals as two-valued at the meta-level, since every world is denoted by the nominal i or it is not (for all $w \in W$, either $w = \bar{i}$ or $w \neq \bar{i}$). An attempt to make the hybrid logic truly many-valued is the logic $\mathbf{MVHL}_5$. Then again, in $\mathbf{MVHL}_5$ nominals suddenly denote sets of worlds, which is a completely new way of looking at hybrid logic. In this direction a lot more research is still to be done.

Even though, many of the presented logics have some undesirable consequences, the diversity of the possible many-valued hybrid logics shows that there are still many questions about hybrid logic that are unanswered. This is even more so, since the underlying modal logic of **MHVL** and $\mathbf{MVHL}_1$-$\mathbf{MVHL}_5$, is just one of several possible ones.

This paper contains a comparison of different many-valued hybrid logics merely based on a semantic analysis. It would be interesting to see how the semantic differences between the logics will be reflected in their proof theory. Can the tableau system of [83] be changed to work for all the logics $\mathbf{MVHL}_1$-$\mathbf{MVHL}_5$ without destroying either decidability or completeness? This is still an open question.

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Chapter 4

A Hybrid Public Announcement Logic with Distributed Knowledge

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Abstract: In this paper the machinery of Hybrid Logic and the logic of public announcements are merged. In order to bring the two logics together properly the underlying hybrid logic has been changed such that nominals only partially denote states. The hybrid logic contains nominals, satisfaction operators, the downarrow binder as well as the global modality. Following this, an axiom system for the Hybrid Public Announcement Logic is presented and using reduction axioms general completeness (in the usual style of Hybrid Logic) is proved. The general completeness allows for an easy way of adding distributed knowledge. Furthermore it turns out that distributed knowledge is definable using satisfaction operators and the downarrow binder. The standard way of adding distributed knowledge using reduction axioms is also discussed and generalized to other modalities sharing properties with the distributed knowledge modality.

Keywords: Hybrid Logic, Public Announcement Logic, Distributed Knowledge, Completeness, Reduction Axioms, Epistemic Logic.

4.1 Introduction

When Arthur Prior introduced Hybrid Logic, it was in the context of temporal logics (see [26]), and since then several applications in temporal logics have been found for Hybrid Logic [29]. However, Hybrid Logic can be viewed as an extension of any kind of modal logic, such as Epistemic Logic. Thus, it is a natural step to extend Epistemic Logic to a hybrid version, but this step has rarely been taken. This paper remedies this insufficiency.

A recent trend in Epistemic Logic is to model the dynamics of knowledge. There are several ways of doing this, and Dynamic Epistemic Logic (DEL) is one type that has received increased attention (see for instance the textbook [163]). The simplest fragment of DEL is Public Announcement Logic (PAL), which adds modalities for the action of public announcement to Epistemic Logic. The main concern of this paper is to combine PAL with Hybrid Logic.

PAL is obtained by adding modalities of the form $[\varphi]$ (for all formulas φ of the language) to the language of Epistemic Logic. The reading of the formula $[\varphi]\psi$ is "after public announcement of φ, ψ is true" and the semantics specify that $[\varphi]\psi$ is true in a state in a model if, and only if, ψ is true at that state in the submodel obtained by restricting the domain to states where φ is true. A central part of Hybrid Logic is the nominals, which are special propositional variables that are interpreted as only being true in one state. In this way we can name and refer to specific states of a model. When combining PAL with Hybrid Logic the immediate problem is that when moving to submodels the states that some nominals name/denote might be removed, and thus conflict with the requirement that nominals must be true in exactly one state. This problem can be overcome by only letting nominals partially denote states. General completeness results from Hybrid Logic can then be transferred to Public Announcement Logic, and this is the first contribution of this paper. A by-product of the general completeness is a straightforward way of adding modal operators such as distributed knowledge to the logic. Indeed adding extra modalities to the language can in many cases be done in a uniform way, this is another contribution of this paper.

Besides this paper, only a handful of other contributions appear to exist on combining Dynamic Epistemic Logic with Hybrid Logic. In the paper [145] all epistemic actions (of full DEL) are internalized. This is done by adding the epistemic actions to the domain of the models, on the same level as epistemic states, and then by using a hybrid language to refer to them. However, in the process of modeling epistemic scenarios, this may result in a blow-up of the models, which must now also contain the epistemic actions. This is not in line with the usual way of using Kripke models, where the states represent different ways the world might be or different states a system might be in. In [134] a public announcement logic with nominals, global modality, modalities for intentions and preferences is introduced. In that paper, to deal with the interplay between nominals and the public announcement operators, the truth condition for nominals is only changed in the updated models. Thus the updated models are not genuine models for the language. We deal with this deficiency in this paper by letting nominals partially denote states in the original model as well; an approach also taken in [80].

In addition to the question of how to combine Hybrid Logic and epistemic modelling, there is the question of the usefulness of Hybrid Logic in epistemic modelling. The usefulness is illustrated by modal logics for games, for instance. [133] introduces a logic with modalities for preferences, knowledge, and intentions as well as the global modality and nominals. It is shown that the notion of Nash equilibrium is definable in this language and that nominals are necessary in this definition (see [133], Fact 5.5.9). In [158] Nash equilibrium is also defined using distributed knowledge, preference modalities and nominals.

Hybrid Logic can also be used to clarify some of the implicit assumptions made when modelling knowledge by Kripke semantics. For instance $@_i\varphi \to K_a @_i\varphi$ is a validity expressing that if φ is true at a state (named by i), then agent a knows this. Furthermore, if the state named by j is accessible from the state named by i all the agents know this, i.e. $@_i\hat{K}_a j \to K_b @_i \hat{K}_a j$ is valid. Finally, all agents know which state every nominal denotes in the sense that if an agent knows he is at the state w and i denotes w then he knows he is at i ($K_a i$). It also means that if an agent does not consider a state named by i possible, then he knows this ($\neg\hat{K}_a i \to K_a \neg\hat{K}_a i$). Thus the hybrid machinery clarifies the implicit assumption that all the agents know exactly what the model looks like. Uncertainty only comes from the fact that they do not necessarily know in which state of the model they are in.

For a hybrid epistemic logic with the downarrow binder $\downarrow x.$[1] we can express that an agent knows all the (relevant) facts at a given state without specifying what they are. The formula $\downarrow x.K_a x$ thus expresses that agent a is completely informed in the current state. This cannot be expressed in basic Epistemic Logic if there are infinitely many propositional symbols, nor if the intended

[1] The intuition behind the operator $\downarrow x.$ is that it names the current state x and by doing so it allows us to return to the state later on.

model is infinite. Imagine a scenario where agent a writes down a natural number (potentially any natural number) and agent b does not see which number. A Kripke model of this scenario will consist of all the natural numbers corresponding to all the possible numbers a could write down. Expressing in classical epistemic logic that agent b knows that a knows what number he writes down would require an infinite disjunction ($K_b(K_a 0 \vee K_a 1 \vee K_a 2 \vee ...)$), where in hybrid logic the formula $K_b \downarrow x.K_a x$ does the trick.

The main focus of this paper is another advantage of introducing Hybrid Logic machinery into PAL. From a proof theoretical point of view, classical Hybrid Logic fixes a great deal of the problems of classical modal logic. In the case of PAL the proof theory also becomes much nicer when we move to a hybrid version, as already demonstrated by the paper [80].

The structure of this paper is as follows: In Section 4.2 Hybrid Logic with Partially Denoting Nominals is introduced and axiomatized. Next, a Hybrid Logic version of PAL is presented, and a sound and complete axiomatization is given (Section 4.3). In Section 4.4 we discuss how distributed knowledge can be added in three different ways. In the process it is shown that distributed knowledge can be defined using satisfaction operators and the downarrow binder. It is also shown how other modalities can be added in a uniform way, generalized from one of the ways distributed knowledge has been added. Finally, concluding remarks and further directions of research are given in Section 4.5.

4.2 A hybrid logic with partial denoting nominals

The basic idea behind letting nominals partially denote states is that they are true in *at most one* state instead of *exactly one* state. But problems arise with the formula $@_i\varphi$, stating that φ is true at the state denoted by i. If the nominal i does not denote a state, what should the truth value of $@_i\varphi$ be? There seems to be only two obvious answers, either $@_i\varphi$ is true in all states or it is false in all states.[2] We choose the second and thus take the formula $@_i\varphi$ to be true if the nominal i denotes a state and φ is true there. The dual operator of $@_i$, denoted by $\overline{@}_i$ (i.e. $\overline{@}_i\varphi := \neg @_i \neg\varphi$), then corresponds to the other choice. The two choices for $@_i\varphi$ make the logic differ from classical Hybrid Logic, since @ is no longer self-dual. Instead the satisfaction operator

[2] If i does not denote any states in a model it does not point out anything else than the empty set, thus it seems only fair to make $@_i\varphi$ true in the entire model or false in the entire model independent of φ.

has been split into an existential modality $@_i$ and a universal modality $\overline{@}_i$.

We will also add the global modality E to the language, where $E\varphi$ is interpreted as "there is some state in the model where φ is true". Since the semantics of this operator do not depend on the nominals, no problem arises by adding this. When adding the modalities E and A (A being the dual of E), the choice of the semantics for $@_i\varphi$ can be seen as the choice between making $@_i\varphi$ equivalent to $E(i \wedge \varphi)$ or $A(i \rightarrow \varphi)$. When nominals only partially denote states these two formulas are no longer equivalent. Since we will have that $@_i\varphi$ is equivalent to $E(i \wedge \varphi)$ and $\overline{@}_i\varphi$ is equivalent to $A(i \rightarrow \varphi)$, we see that the satisfaction operator has been split into an existential modality $@_i$ and a universal modality $\overline{@}_i$.

Besides the global modality we will also add the downarrow binder. Thus we add formulas of the form $\downarrow x.\varphi$ to the language, having the intuitive reading "naming the current state x makes φ true". In adding $\downarrow x.$, we also allow x and $@_x\varphi$ to occur as formulas and we are thus faced with the same problems of denotation. However, now the denotation of a state variable as x is taken care of by assignments and not by the model. Hence we now have to allow partial functions as assignments.

4.2.1 Syntax and semantics

To define the language, we assume a set of propositional variables PROP, a countable infinite set of nominals NOM, and a countable infinite set of state variables SVAR. Since the enterprise is Epistemic Logic, we will denote the modal box operators by K_a, where a is an agent from a finite set $\mathbb{A}$ of agents. (Thus, we are defining a multi-modal logic.)

Definition 33. *The syntax of the full language of Hybrid Logic with Partially Denoting Nominals, denoted by $\mathcal{PH}(@, \downarrow, E)$, is given by*

$$\varphi ::= p \mid u \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K_a\varphi \mid @_u\varphi \mid \downarrow x.\varphi \mid E\varphi,$$

where $p \in \mathsf{PROP}$, $u \in \mathsf{NOM} \cup \mathsf{SVAR}$, $x \in \mathsf{SVAR}$ and $a \in \mathbb{A}$.[3]

We will also be interested in sub-languages of this full language. The language without the global modality E will be denoted by $\mathcal{PH}(@, \downarrow)$ and if we also omit the downarrow binder (and thus also omit the cases for the state variable x) we will denote the language by $\mathcal{PH}(@)$. Finally this language

[3] In the following we will use i, j, k to range over nominals, x, y to range over state variables, and u, s, t to range over both nominals and state variables.

added the global modality will be denoted by $\mathcal{PH}(@, E)$.[4] Furthermore we will use the following abbreviations of $\overline{@}_i$ for $\neg @_i \neg$ and $\hat{K}_a$ for $\neg K_a \neg$.

These languages do not differ from classical Hybrid Logic in the syntax, but their semantics differ. The notion of a frame is the usual one; a frame is a pair $\langle W, (R_a)_{a \in \mathbb{A}} \rangle$ such that R_a is a binary relation on the non-empty set W.[5] Given a frame we can build a model upon it and define truth relative to it.

Definition 34. *Given a frame $\langle W, (R_a)_{a \in \mathbb{A}} \rangle$, a model based upon it is a tuple $\mathcal{M} = \langle W, (R_a)_{a \in \mathbb{A}}, V \rangle$, such that $V : \mathsf{PROP} \cup \mathsf{NOM} \to \mathcal{P}(W)$ satisfies that $|V(i)| \leq 1$, for all $i \in \mathsf{NOM}$. An assignment in $\mathcal{M}$ is a partial function $g : \mathsf{SVAR} \to W$. (By "$x \in dom(g)$" we will denote that x is in the domain of the partial function g.)*

Definition 35. *Let $\mathcal{M} = \langle W, (R_a)_{a \in \mathbb{A}}, V \rangle$ be a model, $w \in W$ and g an assignment in $\mathcal{M}$. The semantics of φ is inductively defined by:*

$\mathcal{M}, w, g \models p$	*iff*	$w \in V(p)$;
$\mathcal{M}, w, g \models i$	*iff*	$w \in V(i)$;
$\mathcal{M}, w, g \models x$	*iff*	$x \in dom(g)$ *and* $g(x) = w$;
$\mathcal{M}, w, g \models \neg\varphi$	*iff*	$\mathcal{M}, w, g \not\models \varphi$;
$\mathcal{M}, w, g \models \varphi \wedge \psi$	*iff*	$\mathcal{M}, w, g \models \varphi$ *and* $\mathcal{M}, w, g \models \psi$;
$\mathcal{M}, w, g \models K_a\varphi$	*iff*	*for all* $v \in W$, *if* wR_av *then* $\mathcal{M}, v, g \models \varphi$;
$\mathcal{M}, w, g \models @_i\varphi$	*iff*	*there is a* $v \in V(i)$ *s.t.* $\mathcal{M}, v, g \models \varphi$;
$\mathcal{M}, w, g \models @_x\varphi$	*iff*	$x \in dom(g)$ *and* $\mathcal{M}, g(x), g \models \varphi$;
$\mathcal{M}, w, g \models \downarrow x.\varphi$	*iff*	$\mathcal{M}, w, g' \models \varphi$, *where* g' *is as* g *besides that* $g'(x) = w$;
$\mathcal{M}, w, g \models E\varphi$	*iff*	*there is a* $v \in W$ *s.t.* $\mathcal{M}, v, g \models \varphi$.

The logic of this semantics will be denoted by $\mathbf{K}_{\mathcal{PH}(@,\downarrow,E)}$ (and similar for the sublanguages). The notions of satisfiability and validity are defined as usual. Note, that if we have a language without the downarrow binder, we do not need assignments, and we will simply omit them.

Some classical validities of Hybrid Logic fail in this new semantics. For instance the formula $@_i i$ is no longer valid. Furthermore $@_i @_j \varphi$ is no longer

[4]As usual in hybrid logic $@_i\varphi$ can be defined as $E(i \wedge \varphi)$, thus the $@_i$ operators are superfluous when we have E. Still, we prefer to keep the $@_i$ operators in the language to make the forthcoming axiomatization more uniform and easier to read.

[5]Note that we do not require that R_a is an equivalence relation as usually done in epistemic logic. However, this requirement can easily be added and will be discussed later on.

equivalent to $@_j\varphi$, however, $@_i@_j\varphi \to @_j\varphi$ remains valid. As already mentioned, self-duality of @ also fails, and this makes the validity $\neg @_i\varphi \leftrightarrow @_i\neg\varphi$ fail. $@_i\neg\varphi \to \neg @_i\varphi$ is valid though and so is $@_i\varphi \to \neg @_i\neg\varphi$, which can been seen as expressing that the satisfaction operator $@_i$ is functional.

Even though $\mathbf{K}_{\mathcal{PH}(@,\downarrow,E)}$ is different from classical Hybrid Logic, we can recover a version of classical Hybrid Logic within $\mathbf{K}_{\mathcal{PH}(@,\downarrow,E)}$. Note that, the formula $@_i i$ (or equivalent Ei) is true exactly when the nominal i denotes a state. Thus putting $@_i i$ as an antecedent to classical hybrid validities will yield validities in $\mathbf{K}_{\mathcal{PH}(@,\downarrow,E)}$, for instance the formulas $@_i i \to (@_j\varphi \leftrightarrow @_i@_j\varphi)$, $@_i i \to (@_i\varphi \leftrightarrow \neg @_i\neg\varphi)$, and $@_i i \to \big(E(i \wedge \varphi) \leftrightarrow A(i \to \varphi)\big)$ become valid. Note also that all classical Hybrid Logic models are models for $\mathbf{K}_{\mathcal{PH}(@,\downarrow,E)}$, thus all validities of $\mathbf{K}_{\mathcal{PH}(@,\downarrow,E)}$ are validities of classical Hybrid Logic.

The validities and equivalences just discussed are used in most proof systems for Hybrid Logic, thus to give a proof system for Hybrid Logic with Partial Denoting Nominals, different axioms and rules are required.

4.2.2 Complete proof systems

We will now give Hilbert-style proof systems for the hybrid logics with partially denoting nominals. We will start by discussing the logic with nominals, satisfaction operators, and downarrow binders $\mathbf{K}_{\mathcal{PH}(@,\downarrow)}$ and completeness for this. Completeness of the logic $\mathbf{K}_{\mathcal{PH}(@)}$ can be obtained in a similar manner. Finally we briefly discuss how the global modality can be added as well as how completeness with respect to other classes of frames can be obtained.

The proof system for $\mathbf{K}_{\mathcal{PH}(@,\downarrow)}$ (and $\mathbf{K}_{\mathcal{PH}(@,\downarrow,E)}$) is shown in Figure 4.1 and follows that of [28] and [8][6], however, some modifications have to be made. Note that, since $@_i$ is a diamond modality we do not have a necessitation rule for $@_i$. However, since $\overline{@}_i$ works as a box modality we get a sound necessitation rule for $\overline{@}_i$. The normal K axiom for @ also gets replaced by a K axiom for $\overline{@}$. Furthermore we loose self duality but can keep "part" of the axiom as $@_i\varphi \to \neg @_i\neg\varphi$. As mentioned before we do not have the full agree axiom in form of $@_i\varphi \leftrightarrow @_j@_i\varphi$, but only the one direction $@_j@_i\varphi \to @_i\varphi$. Reflexivity has also been weakened to $\overline{@}_i i$. Additionally in the $Name$ rule, a $@_i$ has been replaced by a $\overline{@}_i$ to keep the rule useful, since nothing of the form $@_i\varphi$ can ever be provable because i does not denote something in every model. Finally, contrary to [28] we have left out a substitution rule. The reason is that the

[6]The proof system of [28] is also given Figure 1.6 of Section 1.2.2

validities of public announcement logic are not closed under substitution[7] and thus when we want to add the public announcement machinery we cannot have a substitution rule. Thus, we have to give up axioms, but we also have to add new ones. The first new axiom *Denote* simply gives the conditions under which a nominal i denotes. If $@_i\varphi$ is true, it must be because i denotes a state and φ is true there, hence i must denote. The other new axiom *Collapse* says that if the nominal i does denote (i.e. $@_i i$ is true) then the $\overline{@}_i$ operator collapses to the $@_i$ operator. At last note that we are working in a multi-modal language with a modality K_a for each $a \in \mathbb{A}$ and thus for axioms and rules involving a modality we have one axiom/rule for each $a \in \mathbb{A}$.

Axioms for $\mathbf{K}_{\mathcal{PH}(@,\downarrow)}$:

All substitution instances of propositional tautologies	
$K_a(\varphi \to \psi) \to (K_a\varphi \to K_a\psi)$	$\mathrm{K}_\Box$
$\overline{@}_u(\varphi \to \psi) \to (\overline{@}_u\varphi \to \overline{@}_u\psi)$	$\mathrm{K}_{\overline{@}}$
$@_u\varphi \to \overline{@}_u\varphi$	@-functional
$\overline{@}_u u$	Weak-reflexivity
$@_u @_s\varphi \to @_s\varphi$	Weak-agree
$u \to (\varphi \leftrightarrow @_u\varphi)$	Introduction
$\hat{K}_a @_u\varphi \to @_u\varphi$	Back
$(@_u\hat{K}_a s \wedge @_s\varphi) \to @_u\hat{K}_a\varphi$	Bridge
$@_u\varphi \to @_u u$	Denote
$@_u u \to (\overline{@}_u\varphi \to @_u\varphi)$	Collapse
$\overline{@}_u(\downarrow x.\varphi \leftrightarrow \varphi[x := u])$[1]	DA

Rules for $\mathbf{K}_{\mathcal{PH}(@,\downarrow)}$:

From φ and $\varphi \to \psi$, infer ψ	Modus ponens
From φ, infer $K_a\varphi$	Necessitation of $\Box$
From φ, infer $\overline{@}_u\varphi$	Necessitation of $\overline{@}$
From $\overline{@}_u\varphi$, where u does not occur in φ, infer φ	Name
From $(@_u\hat{K}_a s \wedge @_s\varphi) \to \psi$, where $u \neq s$ and s does not occur in φ or ψ, infer $@_u\hat{K}_a\varphi \to \psi$	Paste

Extra axioms for $\mathbf{K}_{\mathcal{PH}(E,-)}$:

$@_i i \to Ei$, (for all $i \in \mathsf{NOM}$)	GM

[1] $\varphi[x := u]$ denotes the formula obtained from φ by substituting all free occurrences of x by u.

Figure 4.1: The Hilbert-style proof systems for $\mathbf{K}_{\mathcal{PH}(@)}$ and its extensions.

[7] $[p]p$ is a validity for all propositional variables p, but $[\varphi]\varphi$ is not a validity for arbitrary formulas φ.

We use the standard terminology for Hilbert-style proof systems. A proof of φ in $\mathbf{K}_{\mathcal{PH}(-)}$ ("$-$" denotes any combination of @, $\downarrow$, and E) is a finite sequence of formulas ending with φ such that every formula in the sequence is either an axiom of $\mathbf{K}_{\mathcal{PH}(-)}$ or follows from previous formulas in the sequence using one of the proof rules. We denote this by $\vdash_{\mathbf{K}_{\mathcal{PH}(-)}} \varphi$. For a set of formulas Γ, $\Gamma \vdash_{\mathbf{K}_{\mathcal{PH}(-)}} \varphi$ holds if there are $\psi_1, ..., \psi_n \in \Gamma$ such that $\vdash_{\mathbf{K}_{\mathcal{PH}(-)}} (\psi_1 \wedge ... \wedge \psi_n) \rightarrow \varphi$. Given a set of formulas Σ, let $\mathbf{K}_{\mathcal{PH}(-)} + \Sigma$ denote the logic obtained from $\mathbf{K}_{\mathcal{PH}(-)}$ by adding all the formulas in Σ as axioms. That φ is provable in the logic $\mathbf{K}_{\mathcal{PH}(-)} + \Sigma$ will be denoted by $\vdash_{\mathbf{K}_{\mathcal{PH}(-)}+\Sigma} \varphi$. A set of formulas Γ is said to be $\mathbf{K}_{\mathcal{PH}(-)} + \Sigma$-inconsistent if $\Gamma \vdash_{\mathbf{K}_{\mathcal{PH}(-)}+\Sigma} \bot$, and $\mathbf{K}_{\mathcal{PH}(-)} + \Sigma$-consistent otherwise. A formula φ is pure if it does not contain any propositional variables or state variables. A set of formulas Σ is called substitution-closed, if it is closed under uniform substitution of nominals for nominals.[8]

4.2.2.1 The completeness proof for $\mathbf{K}_{\mathcal{PH}(@,\downarrow)}$

We start out by stating a Lindenbaum lemma.

Lemma 36 (Lindenbaum lemma)**.** *Let Σ be a set of pure $\mathcal{PH}(@,\downarrow)$-formulas. Every $\mathbf{K}_{\mathcal{PH}(@,\downarrow)} + \Sigma$-consistent set of formulas Γ can be extended to a maximal $\mathbf{K}_{\mathcal{PH}(@,\downarrow)} + \Sigma$-consistent set Γ^+ (in a new language obtained by adding countable many new nominals), such that*

(1) Γ^+ contains a nominal.

(2) For all $@_u \hat{K}_a \varphi \in \Gamma^+$ there is a nominal j, such that $@_u \hat{K}_a j \in \Gamma^+$ and $@_j \varphi \in \Gamma^+$.

Proof. Let Σ and Γ be given as in the lemma. Extend the language with a countable infinite set of new nominals (thus we have infinitely many nominals not occurring in Γ). Enumerate the countably many formulas of this extended language as $(\varphi_n)_{n \in \mathbb{N}}$.

Let $\Gamma_0 = \Gamma \cup \{i_0\}$ for a nominal i_0 not occurring in Γ. Now Γ_0 is consistent (in the rest of this subsection consistent means $\mathbf{K}_{\mathcal{PH}(@,\downarrow)} + \Sigma$-consistent), for assume otherwise: Then there are $\psi_1, ..., \psi_m \in \Gamma$ such that $\vdash_{\mathbf{K}_{\mathcal{PH}(@,\downarrow)}+\Sigma} (i_0 \wedge \psi_1 \wedge ... \wedge \psi_m) \rightarrow \bot$, hence $\vdash_{\mathbf{K}_{\mathcal{PH}(@,\downarrow)}+\Sigma} i_0 \rightarrow ((\psi_1 \wedge ... \wedge \psi_m) \rightarrow \bot)$. Using the $K_{\overline{@}}$-axiom and necessitation of $\overline{@}$ we get $\vdash_{\mathbf{K}_{\mathcal{PH}(@,\downarrow)}+\Sigma} \overline{@}_{i_0} i_0 \rightarrow \overline{@}_{i_0}((\psi_1 \wedge$

[8] For instance, if Σ is substitution-closed and $@_i(p \rightarrow (j \wedge K_a j)) \in \Sigma$ then also $@_k(p \rightarrow (l \wedge K_a l)) \in \Sigma$ for all nominals k and l.

$... \wedge \psi_m) \to \bot)$. Then using weak-reflexivity and modus ponens it follows that $\vdash_{\mathbf{K}_{\mathcal{PH}(@,\downarrow)}+\Sigma} \overline{@}_{i_0}((\psi_1 \wedge ... \wedge \psi_m) \to \bot)$. Finally since i_0 did not occur in Γ we get from the Name rule that $\vdash_{\mathbf{K}_{\mathcal{PH}(@)}+\Sigma} (\psi_1 \wedge ... \wedge \psi_m) \to \bot$, which is a contradiction since Γ is assumed to be consistent. Hence Γ_0 must also be consistent.

Now for $n \in \mathbb{N}$ we define Γ_n in the following way:

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\varphi_n\}, & \text{if } \varphi_n \text{ is not of the form } @_u\hat{K}_a\psi \text{ and} \\ & \text{the set } \Gamma_n \cup \{\varphi_n\} \text{ is consistent.} \\ \Gamma_n \cup \{\varphi_n, @_u\hat{K}_a j, @_j\psi\}, & \text{if } \varphi_n \text{ is of the form } @_u\hat{K}_a\psi,\ j \text{ is a new} \\ & \text{nominal not occurring in } \Gamma_n \text{ or } \varphi_n, \\ & \text{and the set } \Gamma_n \cup \{\varphi_n\} \text{ is consistent.} \\ \Gamma_n, & \text{otherwise.} \end{cases}$$

Then Γ_n is consistent for all $n \in \mathbb{N}$. The proof of this goes by induction on $n \in \mathbb{N}$ and the start has just been shown for Γ_0. The only non-trivial case in the induction step is the case where φ_n is on the form $@_u\hat{K}_a\psi$ and $\Gamma_n \cup \{\varphi_n\}$ is consistent. Assume toward a contradiction that $\Gamma_{n+1} = \Gamma_n \cup \{\varphi_n, @_u\hat{K}_a j, @_j\psi\}$ is inconsistent. Then there are $\psi_1, ..., \psi_m \in \Gamma_n$ such that $\vdash_{\mathbf{K}_{\mathcal{PH}(@,\downarrow)}+\Sigma} (\varphi_n \wedge @_u\hat{K}_a j \wedge @_j\psi \wedge \psi_1 \wedge ... \wedge \psi_m) \to \bot$, thus $\vdash_{\mathbf{K}_{\mathcal{PH}(@,\downarrow)}+\Sigma} (@_u\hat{K}_a j \wedge @_j\psi) \to (\varphi_n \to ((\psi_1 \wedge ... \wedge \psi_m) \to \bot))$. But then since j is new to φ_n and Γ_n it follows from the paste rule that $\vdash_{\mathbf{K}_{\mathcal{PH}(@,\downarrow)}+\Sigma} @_u\hat{K}_a\psi \to (\varphi_n \to ((\psi_1 \wedge ... \wedge \psi_m) \to \bot))$, i.e. $\vdash_{\mathbf{K}_{\mathcal{PH}(@,\downarrow)}+\Sigma} \varphi_n \to ((\psi_1 \wedge ... \wedge \psi_m) \to \bot)$. This is of course a contradiction to the assumption of $\Gamma_n \cup \{\varphi_n\}$ being consistent. Hence Γ_{n+1} must be consistent and it follows by induction that Γ_n is consistent for all $n \in \mathbb{N}$.

Now it easily follows that $\Gamma^+ := \bigcup_{n\in\mathbb{N}} \Gamma_n$ is also consistent. That Γ^+ contains a nominal follows from the construction of $\Gamma_0 = \Gamma \cup \{i_0\}$. And finally the last property follows from the construction of Γ_{n+1} in the case where φ is on the form $@_i\hat{K}_a\psi$. This completes the proof. □

Before we go on to the completeness proof a small lemma is needed.

Lemma 37. *The following are derivable in the logic* $\mathbf{K}_{\mathcal{PH}(@,\downarrow)}$*:*

i) $@_u s \to (\overline{@}_u\varphi \leftrightarrow @_u\varphi)$

ii) $@_u s \to @_s u$

iii) $(@_u u \wedge @_s s) \to (@_s\varphi \leftrightarrow @_u @_s\varphi)$

iv) $@_u s \to (@_u\varphi \leftrightarrow @_s\varphi)$

v) $(@_u s \wedge @_s t) \to @_u t$

Proof. The following are Hilbert style derivations of the formulas:

Proof of i):

(1)	$@_u\varphi \to \overline{@}_u\varphi$	@-functional
(2)	$@_u s \to (@_u\varphi \to \overline{@}_u\varphi)$	Prop. logic on (1)
(3)	$@_u s \to @_u u$	Denote
(4)	$@_u u \to (\overline{@}_u\varphi \to @_u\varphi)$	Collapse
(5)	$@_u s \to (\overline{@}_u\varphi \leftrightarrow @_u\varphi)$	Prop. logic on (2), (3) and (4)

Proof of ii):

(1)	$s \to (u \to @_s u)$	Introduction
(2)	$\overline{@}_u s \to (\overline{@}_u u \to \overline{@}_u @_s u)$	Necessitation of $\overline{@}$ and $K_{\overline{@}_u}$ on (1)
(3)	$\overline{@}_u s \to \overline{@}_u @_s u$	Weak-reflexivity and prop. logic on (2)
(4)	$@_u s \to \overline{@}_u @_s u$	@-functional and prop. logic on (3)
(5)	$@_u s \to @_u @_s u$	i) and prop. logic on (4)
(6)	$@_u s \to @_s u$	Weak-agree and prop. logic on (5)

Proof of iii):

(1)	$@_u @_s \neg\varphi \to @_s \neg\varphi$	Weak-agree
(2)	$\neg @_s \neg\varphi \to \neg @_u @_s \neg\varphi$	Prop.logic on (1)
(3)	$\overline{@}_s\varphi \to \overline{@}_u \overline{@}_s\varphi$	Definition of $\overline{@}$ on (2)
(4)	$@_s s \to (\overline{@}_s\varphi \leftrightarrow @_s\varphi)$	i)
(5)	$(\varphi \leftrightarrow \psi) \to (\overline{@}_u\varphi \leftrightarrow \overline{@}_u\psi)$	Prop. logic and nec. of $\overline{@}$ and $K_{\overline{@}}$
(6)	$@_s s \to (@_s\varphi \to \overline{@}_u @_s\varphi)$	Prop. logic on (3), (4) and (5)
(7)	$@_u u \to (\overline{@}_u\varphi \leftrightarrow @_u\varphi)$	i)
(8)	$(@_u u \wedge @_s s) \to (@_s\varphi \to @_u @_s\varphi)$	(6), (7) and prop. logic
(9)	$(@_u u \wedge @_s s) \to (@_u @_s\varphi \to @_s\varphi)$	Weak-agree and prop. logic.
(10)	$(@_u u \wedge @_s s) \to (@_s\varphi \leftrightarrow @_u @_s\varphi)$	Prop. logic on (8) and (9)

Proof of *iv*):

(1)	$s \to (\varphi \leftrightarrow @_s\varphi)$	Introduction
(2)	$\overline{@}_u s \to (\overline{@}_u\varphi \leftrightarrow \overline{@}_u @_s\varphi)$	Necessitation of $\overline{@}$ and $K_{\overline{@}_u}$ on (1)
(3)	$@_u s \to (\overline{@}_u\varphi \leftrightarrow \overline{@}_u @_s\varphi)$	@-functional and prop. logic on (2)
(4)	$@_u s \to (@_u\varphi \leftrightarrow @_u @_s\varphi)$	*i*) and prop. logic on (3)
(5)	$@_u s \to (@_u\varphi \leftrightarrow @_s\varphi)$	*iii*), *ii*), Denote and prop. logic on (4)

Proof of *v*):

(1)	$@_u s \to (@_u t \leftrightarrow @_s t)$	*iv*)
(2)	$(@_u s \wedge @_s t) \to @_u t$	Prop. logic on (1)

□

With this lemma, we can now construct a Henkin style model.

Definition 38. *Let Γ be a maximal consistent set of $\mathcal{PH}(@, \downarrow)$-formulas. Define $\mathcal{N}_\Gamma = \{u \in \mathsf{NOM} \cup \mathsf{SVAR} \mid @_u u \in \Gamma\}$ and an equivalence relation $\sim$ on $\mathcal{N}_\Gamma$ by $u \sim s$ iff $@_u s \in \Gamma$ (and denote the equivalence class of u by $|u|$). Then the canonical model $\mathfrak{M}_\Gamma = \langle W, (R_a)_{a\in\mathbb{A}}, V\rangle$ and the canonical assignment g_Γ are defined by*

$$\begin{array}{rcl} W &=& \{|u| \mid u \in \mathcal{N}_\Gamma\}; \\ |u|R_a|s| & \textit{iff} & @_u\hat{K}_a s \in \Gamma \quad \textit{for all } a \in \mathbb{A}; \\ V(p) &=& \{|u| \in W \mid @_u p \in \Gamma\} \quad \textit{for all } p \in \mathsf{PROP}; \\ V(j) &=& \{|u| \in W \mid @_u j \in \Gamma\} \quad \textit{for all } j \in \mathsf{NOM}; \\ g_\Gamma(x) &=& |x| \quad \textit{for all } x \in \mathsf{SVAR} \cap \mathcal{N}_\Gamma. \end{array}$$

A few comments about why this is well-defined are in order. First of all note that by the Denote rule and *ii*) of Lemma 37, if $@_u s \in \Gamma$ then $u, s \in \mathcal{N}_\Gamma$. That the relation $\sim$ is an equivalence relation (and thus W is well-defined) follows from the construction of $\mathcal{N}_\Gamma$ and *ii*) and *v*) of Lemma 37. That R_a is well-defined follows from *iv*) of Lemma 37 and the Bridge axiom. Finally that V is well-defined for $p \in \mathsf{PROP}$ follows from *iv*) of Lemma 37, and for $i \in \mathsf{NOM}$ by $\sim$ being an equivalence relation. $\sim$ being an equivalence relation also guaranties that g_Γ is a well-defined assignment. Note that if $@_i i \notin \Gamma$ then $V(i) = \emptyset$ and thus i does not denote. Similar for state variables.

An essential part of the completeness proof is the following truth lemma:

Lemma 39 (Truth lemma)**.** *Let Γ be a maximal consistent set of $\mathcal{PH}(@,\downarrow)$-formulas that satisfy item* (2) *of the Lindenbaum lemma. Then for all $u \in \mathcal{N}_\Gamma$ and all $\mathcal{PH}(@,\downarrow)$-formulas φ:*

$$\mathfrak{M}_\Gamma, |u|, g_\Gamma \models \varphi \quad \textit{iff} \quad @_u\varphi \in \Gamma. \tag{4.1}$$

Proof. The proof goes by induction on φ. When φ is a p or j for a $p \in \mathsf{PROP}$ or $j \in \mathsf{NOM}$, (4.1) follows directly from the definition of V. When φ is on the form x for a $x \in \mathsf{SVAR}$, (4.1) follows from $\sim$ being an equivalence relation. This takes care of the induction basis.

The induction step. In the case φ is on the form $\psi\wedge\chi$, note that $@_u\psi, @_u\chi \in \Gamma$ if and only if $@_u(\psi \wedge \chi) \in \Gamma$, which can be proved using propositional logic and the rules and axioms Denote, Collapse, $K_{\overline{@}}$ and necessitation of $\overline{@}$. In the case φ is on the form $\neg\psi$, the thing to note is that $\neg @_u\psi \in \Gamma \Leftrightarrow @_u\neg\psi \in \Gamma$. "$\Leftarrow$" follows from the axiom @-functional. "$\Rightarrow$" follows using Collapse and the fact that $@_u u \in \Gamma$ by the assumption $u \in \mathcal{N}_\Gamma$.

Assume now that φ has the form $@_s\psi$. First note that if $@_u@_s\psi \in \Gamma$ then $@_s\psi \in \Gamma$ by weak-agree and thus $s \in \mathcal{N}_\Gamma$ by the Denote axiom. Then by induction it follows that $\mathcal{M}_\Gamma, |s|, g_\Gamma \models \psi$, which again implies that $\mathcal{M}_\Gamma, |u|, g_\Gamma \models @_s\psi$. If $\mathcal{M}_\Gamma, |u|, g_\Gamma \models @_s\psi$ then there is a $s' \in \mathcal{N}_\Gamma$ such that $\mathcal{M}_\Gamma, |s'|, g_\Gamma \models \psi$ and $V(s) = |s'|$ if s is a nominal and $g_\Gamma(s) = |s'|$ if s is a state variable. By the definition of V and g_Γ this implies that $@_{s'}s \in \Gamma$ and by the induction hypothesis that $@_{s'}\psi \in \Gamma$. But now it follows from *iv*) of Lemma 37 that $@_s\psi \in \Gamma$. From the assumption about i and $@_{s'}s \in \Gamma$ and Lemma 37 *ii*) and Denote it follows that $@_u u, @_s s \in \Gamma$. But then by *iii*) of Lemma 37, $@_u@_s\psi \in \Gamma$ follows.

The case φ is of the form $\hat{K}_a\psi$. If $\mathcal{M}_\Gamma, |u|, g_\Gamma \models \hat{K}_a\psi$, then there is a $s \in \mathcal{N}_\Gamma$ such that $|u|R_a|s|$ and $\mathcal{M}_\Gamma, |s|, g_\Gamma \models \psi$. By definition of R_a, $@_u\hat{K}_a s \in \Gamma$ and by the induction hypothesis $@_s\psi \in \Gamma$. But then by the bridge axiom it follows that $@_u\hat{K}_a\psi \in \Gamma$. Now assume that $@_u\hat{K}_a\psi \in \Gamma$. Then since Γ satisfies item (2) of the Lindenbaum lemma it follows that there is a nominal j such that $@_u\hat{K}_a j \in \Gamma$ and $@_j\psi \in \Gamma$. Note that by Denote $j \in \mathcal{N}_\Gamma$. Now by the definition of R_a and V and the induction hypothesis it follows that $\mathcal{M}_\Gamma, |u|, g_\Gamma \models \hat{K}_a\psi$.

Finally for the case where φ is of the form $\downarrow x.\psi$. First note that $\mathcal{M}_\Gamma, |u|, g_\Gamma \models \downarrow x.\psi$ if and only if $\mathcal{M}_\Gamma, |u|, g_\Gamma \models \psi[x := u]$ due to a substitution lemma that can easily be proven.[9] But then by the induction hypothesis it follows that $\mathcal{M}_\Gamma, |u|, g_\Gamma \models \downarrow x.\psi$ if and only if $@_u\psi[x := u] \in \Gamma$. And finally by the DA

[9] Let $Int(u)$ stand for $g(u)$ if u is a state variable and $V(u)$ if u is a nominal. Then

axiom it follows that $\mathcal{M}_\Gamma, |u|, g_\Gamma \models \downarrow x.\psi$ if and only if $@_u \downarrow x.\psi \in \Gamma$. This concludes the proof. $\square$

A frame $\mathcal{F}$ *validates* a set of formulas Σ, if $\mathcal{M} \models \Sigma$ for all models $\mathcal{M}$ based on $\mathcal{F}$. With this notion we state a Frame lemma:

Lemma 40 (Frame lemma). *Let Σ be a substitution-closed set of pure $\mathcal{PH}(@,\downarrow)$-formulas and let Γ be a $\mathbf{K}_{\mathcal{PH}(@,\downarrow)} + \Sigma$ maximal consistent set of $\mathcal{PH}(@,\downarrow)$-formulas satisfying item* (1) *and* (2) *of the Lindenbaum lemma. Then the underlying frame of $\mathfrak{M}_\Gamma$ validates all the formulas in Σ.*

Proof. See Lemma 7.1 of [25]. $\square$

We are now finally capable of proving the completeness theorem.

Theorem 41 (Completeness of $\mathbf{K}_{\mathcal{PH}(@,\downarrow)}$). *Let Σ be a substitution-closed set of pure $\mathcal{PH}(@,\downarrow)$-formulas. Every set of $\mathcal{PH}(@,\downarrow)$-formulas that is $\mathbf{K}_{\mathcal{PH}(@,\downarrow)}+$ Σ-consistent is satisfiable in a model whose underlying frame validates all the formulas in Σ.*

Proof. Assume that Γ is $\mathbf{K}_{\mathcal{PH}(@,\downarrow)} + \Sigma$-consistent. Then it can be extended to a maximal $\mathbf{K}_{\mathcal{PH}(@,\downarrow)} + \Sigma$-consistent set Γ^+ by the Lindenbaum lemma. Since there is a nominal $i \in \Gamma^+$ by item (1) of the Lindenbaum lemma it is easy to see that for all $\varphi \in \Gamma$, $@_i\varphi \in \Gamma^+$ by the Introduction axiom. But then by the truth lemma it follows that $\mathfrak{M}_{\Gamma^+}, |i|, g_{\Gamma^+} \models \Gamma$. By the frame lemma the underlying frame of $\mathfrak{M}_{\Gamma^+}$ validates all the formulas in Σ and the proof is done. $\square$

4.2.2.2 Completeness for $\mathbf{K}_{\mathcal{PH}(@,E,-)}$ and completeness with respect to other frame classes

In the case of completeness with respect to the global modality E, we once more follow the lines of [28]. We take one of the modalities in our multi-modal logic to be E[10] and add the axiom GM of Figure 4.1. To see why this suffices note that E is just a normal modal operator for which the intended accessibility relation is the universal relation on the domain. The formula $@_i i \to Ei$ is

the substitution lemma can be stated as: *Let $\mathcal{M} = \langle W, R, V\rangle$ be a model, φ a formula, and $u \in \mathsf{SVAR} \cup \mathsf{NOM}$. Then for all $w \in W$ and all assignments g with $g(x) = Int(u)$: $\mathcal{M}, w, g \models \varphi$ iff $\mathcal{M}, w, g \models \varphi[x := u]$.*

[10]Elaborated, we add an extra agent e to $\mathbb{A}$ and write E instead of $\hat{K}_e$. Thus, in the proof system we also include all the axioms and rules from Figure 4.1 involving K_a, for E.

a pure formula, so adding all substitution instances, as in the axiom GM, automatically gives completeness with respect to the class of frames $@_i i \to Ei$ defines. Hence, all that is left to notice is that $@_i i \to Ei$ defines the universal relation on the domain. However, this can easily be proven and we obtain:

Theorem 42 (Completeness of $\mathbf{K}_{\mathcal{PH}(@,E,-)}$). *Let Σ be a substitution-closed set of pure $\mathcal{PH}(@,E,-)$-formulas. Every set of $\mathcal{PH}(@,E,-)$-formulas that is $\mathbf{K}_{\mathcal{PH}(@,E,-)} + \Sigma$-consistent is satisfiable in a model whose underlying frame validates all the formulas in Σ.*

In Epistemic Logic one usually wants to put extra conditions on the relations R_a, for instance transitivity, reflexivity, and euclideaness. The logic obtained by requiring all these properties will be denoted $\mathbf{S5}_{\mathcal{PH}(-)}$ and if only transitivity and reflexivity are required, the logic will be denoted by $\mathbf{S4}_{\mathcal{PH}(-)}$. When modal logic is used to reason about beliefs, one usually replaces the reflexivity requirement of $\mathbf{S5}_{\mathcal{PH}(-)}$ by requiring seriality of R_a instead, and the logic obtained in this way will be denoted $\mathbf{KD45}_{\mathcal{PH}(-)}$. Now if one wants to work with these logics instead of just $\mathbf{K}_{\mathcal{PH}(-)}$, complete Hilbert-style proof systems can easily be obtained from theorems 41 and 42, since all the properties can be defined by pure formulas. $i \to \hat{K}_a i$ defines reflexivity, $\hat{K}_a \hat{K}_a i \to \hat{K}_a i$ defines transitivity, $\hat{K}_a i \to K_a \hat{K}_a i$ defines euclideaness, and $\hat{K}_a \top$ defines seriality, which is all well known in the Hybrid Logic literature.

4.3 Hybrid Public Announcement Logic

We now combine Hybrid Logic with Partially Denoting Nominals with PAL. As before we assume the sets PROP, NOM and SVAR, and $\mathbb{A}$. The full language $\mathcal{HPAL}(@,\downarrow,E)$ of the Hybrid Public Announcement Logic is given by:

$$\varphi ::= p \mid u \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K_a\varphi \mid @_u\varphi \mid {\downarrow}x.\varphi \mid E\varphi \mid [\varphi]\varphi,$$

where $p \in \mathsf{PROP}$, $u \in \mathsf{NOM} \cup \mathsf{SVAR}$, $x \in \mathsf{SVAR}$, and $a \in \mathbb{A}$. For the sublanguages we will use the same conventions as before.

The notion of a model $\mathcal{M} = \langle W, (R_a)_{a\in\mathbb{A}}, V\rangle$ is the same as for $\mathcal{PH}(@,\downarrow,E)$. The definition of the semantic entailment $\mathcal{M}, w, g \models \varphi$ is a combination of Definition 35 for $\mathcal{PH}(@,\downarrow,E)$ and the following clause:

$$\mathcal{M}, w, g \models [\varphi]\psi \iff \mathcal{M}, w, g \models \varphi \text{ implies that } \mathcal{M}|_\varphi, w, g_\varphi \models \psi,$$

where the definition of the model $\mathcal{M}|_\varphi = \langle W|_\varphi, R|_\varphi, V|_\varphi \rangle$ is:

$$\begin{array}{lcl} W|_\varphi & = & \{v \in W \mid \mathcal{M}, v, g \models \varphi\} \\ R_a|_\varphi & = & R_a \cap (W|_\varphi \times W|_\varphi) \\ V|_\varphi(p) & = & V(p) \cap W|_\varphi \\ V|_\varphi(i) & = & V(i) \cap W|_\varphi, \end{array}$$

and the assignment g_φ is obtained from g by restricting its domain to the set $\{x \in dom(g) \mid g(x) \in W|_\varphi\}$.

The logic of this semantics will be called the full Hybrid Public Announcement Logic and will be denoted by $\mathbf{K}_{\mathcal{HPAL}(@,\downarrow,E)}$. Note that $\mathcal{M}|_\varphi$ is just the model $\mathcal{M}$ restricted to the states where φ is true. The problem of adding nominals to PAL now becomes immediately clear: If a nominal i denotes a state where φ is not true, i does not denote any state in the model $\mathcal{M}|_\varphi$. The problem arises for state variables as well. This is the main reason for introducing a Hybrid Logic with partially denoting nominals in this paper.[11]

We will provide the logic with a Hilbert-style proof system and show completeness in the usual way for PAL, i.e we will provide a truth-preserving translation from $\mathbf{K}_{\mathcal{HPAL}(@,\downarrow,E)}$ into $\mathbf{K}_{\mathcal{PH}(@,\downarrow,E)}$. This is interesting in its own right, since it shows that Hybrid Public Announcement Logic is not more expressive than the underlying hybrid epistemic logic (which is also the case in standard PAL, see [163]). The proof system is given in Figure 4.2 and is an extension of the one for $\mathbf{K}_{\mathcal{PH}(@,\downarrow,E)}$ with additional reduction axioms for the public announcement operator. These reduction axioms are the usual ones from PAL plus new ones for the hybrid operators.

Before discussing soundness and completeness of the proof system, we give a few comments on the choice of reduction axioms for the new components. For the *Announcement and satisfaction* axiom, the intuition behind it is: If ψ is true at the state u after an announcement of φ, this amounts first of all to the state u remaining in the restricted model, i.e. φ is true at u, and

[11]There is another way of defining the semantics for the public announcement operator $[\varphi]$. Instead of removing states where φ is not true, one simply removes access to these states, i.e. restrict the accessibility relations. In standard PAL these approaches are equivalent, but in Hybrid Logic using either satisfaction operators or the global modality, we are capable of reaching states which are not accessible via the accessibility relations and thus the two approaches differ. In the case of Hybrid Logic the approach of only deleting accessibility relations may seem more appealing since the problem of losing denotation of the nominals is not present anymore. However, there are other drawbacks, which to the author's opinion makes the approach with partially denoting nominals much more appealing. For more on these issues see Section 4.6 of the appendix.

Axioms for $\mathbf{K}_{\mathcal{HPAL}(@,\downarrow,E)}$:

All axioms for $\mathbf{K}_{\mathcal{PH}(@,\downarrow,E)}$

$[\varphi]\, p \leftrightarrow (\varphi \to p)$	Atomic permanence (propositions)
$[\varphi]\, u \leftrightarrow (\varphi \to u)^1$	Atomic permanence (states)
$[\varphi]\, \neg\psi \leftrightarrow (\varphi \to \neg[\varphi]\psi)$	Announcement and negation
$[\varphi]\, (\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$	Announcement and conjunction
$[\varphi]\, K_a\psi \leftrightarrow (\varphi \to K_a[\varphi]\psi)$	Announcement and knowledge
$[\varphi]\, [\psi]\chi \leftrightarrow [\varphi \wedge [\varphi]\psi]\chi$	Announcement composition
$[\varphi]\, @_u\psi \leftrightarrow (\varphi \to @_u(\varphi \wedge [\varphi]\psi))^1$	Announcement and satisfaction
$[\varphi]{\downarrow}x.\psi \leftrightarrow {\downarrow}x.[\varphi]\psi^2$	Announcement and downarrow
$[\varphi]\, E\psi \leftrightarrow (\varphi \to E(\varphi \wedge [\varphi]\psi))$	Announcement and global modality

Rules for $\mathbf{K}_{\mathcal{HPAL}(@,\downarrow,E)}$:

All rules for $\mathbf{K}_{\mathcal{PH}(@,\downarrow,E)}$

[1] Here $u \in \mathsf{NOM} \cup \mathsf{SVAR}$.

[2] Assuming that x does not occur in φ.

Figure 4.2: The Hilbert-style proof system for $\mathbf{K}_{\mathcal{HPAL}(@,\downarrow,E)}$.

secondly to the public announcement of φ at u leading to ψ being true. For the *Announcement and global modality* axiom, almost the same intuition applies. For the downarrow binder a little care has to be taken regarding the reduction axiom. Note that moving a ${\downarrow}x.$-operator from within the scope of a $[\varphi]$-operator outside the scope, can lead to accidental binding of a state variable x in $[\varphi]$, and this might affect the truth value of the formula. Hence the requirement in the *Announcement and downarrow* axiom. However, this is not really a limitation because we can always rename bound variables without changing the truth value of a formula. Thus when encountering a formula $[\varphi]{\downarrow}x.\psi$ where x appears in φ, we can simply replace all occurrences of x in ψ by a new state variable y to get ψ' and obtain an equivalent formula $[\varphi]{\downarrow}y.\psi'$, where y does not occur in φ. With this assumption the reduction axiom for the downarrow binder is sound. The soundness of the reduction axioms for the satisfaction operator, the global modality and the downarrow binder is stated in the following lemma:

Lemma 43. *The following holds for all $\mathcal{HPAL}(@,\downarrow,E)$ formulas φ and ψ:*

1) *$[\varphi]@_u\psi$ is equivalent to $\varphi \to @_u(\varphi \wedge [\varphi]\psi)$.*
2) *$[\varphi]E\psi$ is equivalent to $\varphi \to E(\varphi \wedge [\varphi]\psi)$.*
3) *If the state variable x does not occur in the formula φ, then $[\varphi]{\downarrow}x.\psi$ is equivalent to ${\downarrow}x.[\varphi]\psi$.*

Proof. **1)** Since $\langle\varphi\rangle\psi$ and $\varphi \wedge [\varphi]\psi$ are equivalent (as in standard PAL), one only needs to show that $[\varphi]@_u\psi$ is equivalent to $\varphi \to @_u\langle\varphi\rangle\psi$. This is shown by the following equivalences:

$$
\begin{array}{ll}
 & \mathcal{M}, w, g \models [\varphi]@_u\psi \\
\text{iff} & \mathcal{M}, w, g \models \varphi \Rightarrow \mathcal{M}|_\varphi, w, g_\varphi \models @_u\psi \\
\text{iff} & \mathcal{M}, w, g \models \varphi \Rightarrow (\exists v \in W|_\varphi \text{ s.t. } \mathcal{M}|_\varphi, v, g_\varphi \models u \wedge \mathcal{M}|_\varphi, v, g_\varphi \models \psi) \\
\text{iff} & \mathcal{M}, w, g \models \varphi \Rightarrow (\exists v \in W \text{ s.t. } \mathcal{M}, v, g \models \varphi \wedge \mathcal{M}, v, g \models u \wedge \mathcal{M}|_\varphi, v, g_\varphi \models \psi) \\
\text{iff} & \mathcal{M}, w, g \models \varphi \Rightarrow (\exists v \in W \text{ s.t. } \mathcal{M}, v, g \models u \wedge \mathcal{M}, v, g \models \langle\varphi\rangle\psi) \\
\text{iff} & \mathcal{M}, w, g \models \varphi \Rightarrow \mathcal{M}, w, g \models @_u\langle\varphi\rangle\psi \\
\text{iff} & \mathcal{M}, w, g \models \varphi \to @_u\langle\varphi\rangle\psi
\end{array}
$$

2) This is similar to 1.

3) Let a model $\mathcal{M} = \langle W, (R_a)_{a\in\mathbb{A}}, V\rangle$, a state $w \in W$ and an assignment g in $\mathcal{M}$ be given. Let also formulas φ and ψ be given such that the state variable x does not occur in φ. Note that since x does not occur in φ, for all assignments h and h' such that they only differs on x, $\mathcal{M}, w, h \models \varphi$ if and only if $\mathcal{M}, w, h' \models \varphi$ (for all models $\mathcal{M}$ and states w). We now have the following equivalences:

$$
\begin{array}{lll}
\mathcal{M}, w, g \models [\varphi]{\downarrow}x.\psi & \text{iff} & \mathcal{M}, w, g \models \varphi \;\Rightarrow\; \mathcal{M}|_\varphi, w, g_\varphi \models{\downarrow}x.\psi \\
 & \text{iff} & \mathcal{M}, w, g \models \varphi \;\Rightarrow\; \mathcal{M}|_\varphi, w, g'_\varphi \models \psi \\
 & \text{iff} & \mathcal{M}, w, g' \models \varphi \;\Rightarrow\; \mathcal{M}|_\varphi, w, g'_\varphi \models \psi \\
 & \text{iff} & \mathcal{M}, w, g' \models [\varphi]\psi \\
 & \text{iff} & \mathcal{M}, w, g \models{\downarrow}x.[\varphi]\psi,
\end{array}
$$

where g' is just like g except that $g'(x) = w$ and g'_φ is just like g_φ except that $g'_\varphi(x) = w$. □

The soundness of the proof system follows from the soundness of $\mathbf{K}_{\mathcal{PH}(@,\downarrow,E)}$ together with the soundness of the reduction axioms. For the completeness of the proof system, we first define a translation t from the language of hybrid public announcement logic into the language of hybrid logic with partially denoting nominals, i.e. $t : \mathcal{HPAL}(@,\downarrow,E) \to \mathcal{PH}(@,\downarrow,E)$. The definition of t is given in Figure 4.3. The restriction in the case for $[\varphi]{\downarrow}x.\psi$ is there to avoid accidental binding of x in φ as mentioned earlier.

Note that the translation is not defined inductively on the usual complexity of a formula. Therefore we cannot prove results regarding t by induction

$t(p)$	$=$	p	$t([\varphi]p)$	$=$	$t(\varphi \to p)$
$t(u)$	$=$	u [1]	$t([\varphi]u)$	$=$	$t(\varphi \to u)$ [1]
$t(\neg\varphi)$	$=$	$\neg t(\varphi)$	$t([\varphi]\neg\psi)$	$=$	$t(\varphi \to \neg[\varphi]\psi)$
$t(\varphi \wedge \psi)$	$=$	$t(\varphi) \wedge t(\psi)$	$t([\varphi]\psi \wedge \chi)$	$=$	$t([\varphi]\psi \wedge [\varphi]\chi)$
$t(K_a\varphi)$	$=$	$K_a t(\varphi)$	$t([\varphi]K_a\psi)$	$=$	$t(\varphi \to K_a[\varphi]\psi)$
$t(@_u\varphi)$	$=$	$@_u t(\varphi)$ [1]	$t([\varphi]@_u\psi)$	$=$	$t(\varphi \to @_u(\varphi \wedge [\varphi]\psi))$ [1]
$t({\downarrow}x.\varphi)$	$=$	${\downarrow}x.t(\varphi)$	$t([\varphi]{\downarrow}x.\psi)$	$=$	$t({\downarrow}x'.[\varphi](\psi[x := x']))$ [2]
$t(E\varphi)$	$=$	$Et(\varphi)$	$t([\varphi]E\psi)$	$=$	$t(\varphi \to E(\varphi \wedge [\varphi]\psi))$
			$t([\varphi][\psi]\chi)$	$=$	$t([\varphi \wedge [\varphi]\psi]\chi)$

[1]Where $u \in \mathsf{NOM} \cup \mathsf{SVAR}$. [2]$x'$ is a new state variable not occurring in φ or ψ.

Figure 4.3: The translation $t : \mathcal{HPAL}(@, \downarrow, E) \to \mathcal{PH}(@, \downarrow, E)$.

on this complexity. However, the complexity of the formula immediately succeeding the public announcement operator decreases trough the translation, and this we can use. A new complexity measure $c : \mathcal{HPAL}(@, \downarrow, E) \to \mathbb{N}$ can be defined such that c decreases for every step of the translation, for instance $c([\varphi]@_i\psi) > c(\varphi \to @_i(\varphi \wedge [\varphi]\psi))$. The details of this are omitted, see [163] or [105]. Using this complexity measure we can easily prove that every formula of hybrid public announcement logic is provably equivalent to its translation:

Lemma 44. *For all $\mathcal{HPAL}(@, \downarrow, E)$ formulas φ,*

$$\vdash_{\mathbf{K}_{\mathcal{HPAL}(@,\downarrow,E)}} \varphi \leftrightarrow t(\varphi).$$

From this lemma together with soundness of the proof system, it follows that all formulas is also semantically equivalent to their translation:

Lemma 45. *For all $\mathcal{HPAL}(@, \downarrow, E)$ formulas φ, all models $\mathcal{M} = \langle W, R, V\rangle$, all $w \in W$, and all assignments g,*

$$\mathcal{M}, w, g \models \varphi \iff \mathcal{M}, w, g \models t(\varphi).$$

Note that translating pure formulas from $\mathcal{HPAL}(@, \downarrow, E)$ results in pure formulas in $\mathcal{PH}(@, \downarrow, E)$. A general completeness result now follows:

Theorem 46 (Completeness for $\mathbf{K}_{\mathcal{HPAL}(@,\downarrow,E)}$)**.** *Let Σ be a substitution-closed set of pure $\mathcal{HPAL}(@, \downarrow, E)$-formulas. Every set of $\mathcal{HPAL}(@, \downarrow, E)$-formulas that is $\mathbf{K}_{\mathcal{HPAL}(@,\downarrow,E)} + \Sigma$-consistent is satisfiable in a model whose underlying frame validates all the formulas in Σ.*

Proof. Assume that Γ is $\mathbf{K}_{\mathcal{HPAL}(@\,,\,\downarrow\,,\,E)} + \Sigma$-consistent. For any set of $\mathcal{HPAL}(@, \downarrow, E)$-formulas X, let $t(X) := \{t(\varphi) \mid \varphi \in X\}$. Then $t(\Gamma)$ is $\mathbf{K}_{\mathcal{PH}(@,\downarrow,E)} + t(\Sigma)$-consistent, for assume otherwise: Then there are $\varphi_1, ..., \varphi_n \in \Gamma$ such that $\vdash_{\mathbf{K}_{\mathcal{PH}(@,\downarrow,E)}+t(\Sigma)} t(\varphi_1 \wedge ... \wedge \varphi_n) \to \bot$. But then also $\vdash_{\mathbf{K}_{\mathcal{HPAL}(@,\downarrow,E)}+\Sigma} t(\varphi_1 \wedge ... \wedge \varphi_n) \to \bot$ (using Lemma 44 on formulas in Σ) and by Lemma 44, $\vdash_{\mathbf{K}_{\mathcal{HPAL}(@,\downarrow,E)}+\Sigma} \varphi_1 \wedge ... \wedge \varphi_n \to \bot$, which is a contradiction to Γ being $\mathbf{K}_{\mathcal{HPAL}(@,\downarrow,E)} + \Sigma$-consistent. Now by Theorem 41 $t(\Gamma)$ is satisfiable in a model $\mathcal{M}$ (which is also a model for $\mathcal{HPAL}(@, \downarrow, E)$), and by Lemma 45 it follows that Γ is also satisfiable in $\mathcal{M}$.

Finally, for all pure formulas $\varphi \in \Sigma$, $t(\varphi)$ is a pure formula. Thus by Theorem 41 the underlying frame of $\mathcal{M}$ validates all of the formulas $t(\varphi) \in t(\Sigma)$. But by Lemma 45 the underlying frame then also validates all $\varphi \in \Sigma$. $\square$

Note that we could have left out any of the operators $\downarrow x.$, E, or both and thus got completeness for any of the weaker logics. Theorem 46 also provides completeness with respect to other classes of frames defined by pure formulas and thus we obtain epistemic public announcement logics such as $\mathbf{S4}_{\mathcal{HPAL}(@,\downarrow,E)}$ and $\mathbf{S5}_{\mathcal{HPAL}(@,\downarrow,E)}$.

4.4 Adding distributed knowledge and other modalities

Often notions of group knowledge are important when modeling knowledge in multi-agent settings. Distributed knowledge is such a notion and we will discuss it in detail. Another is common knowledge which we will also shortly mention. We will add distributed knowledge to $\mathbf{K}_{\mathcal{HPAL}(-)}$ in three different ways. The first way is the standard one for public announcement logic; we add distributed knowledge to the underlying logic $\mathbf{K}_{\mathcal{PH}(-)}$ and then give a sound reduction axiom for distributed knowledge. Due to the generality of Theorem 46 we also have another way of adding distributed knowledge; using pure formulas we can add distributed knowledge directly to $\mathbf{K}_{\mathcal{HPAL}(-)}$ getting the reduction axiom for free. The third way only works for extensions of logics that contain satisfaction operators and the downarrow binder. In these logics distributed knowledge becomes directly definable. The first way is a little more involved compared to the other two, but we included it here because we want to generalize this method to other modalities, which also give insight into why common knowledge cannot be added to public announcement logic using reduction axioms.

To add distributed knowledge we add to the given language a modal operator D_B for every non-empty subset $B \subseteq \mathbb{A}$. The semantics of the distributed knowledge operator is:

$$\mathcal{M}, w, g \models D_B\varphi \quad \text{iff} \quad \text{for all } v \in W;\ \text{if } (w,v) \in \bigcap_{b\in B} R_b \text{ then } \mathcal{M}, v, g \models \varphi.$$

The dual operator of D_B will be denoted by $\hat{D}_B$. Note that the semantics of D_B is given in terms of intersection of relations, which in PDL is not modally definable though it is axiomatizable.[12] However, with nominals intersection becomes easy to modally define (see [126] for more on these issues).

4.4.1 Adding distributed knowledge the standard way

Axioms for $\mathbf{K}_{\mathcal{PH}(-,D)}$:
All the axioms for $\mathbf{K}_{\mathcal{PH}(-)}$
All the axioms of $\mathbf{K}_{\mathcal{PH}(-)}$ involving K_a, with K_a replaced by D_B
(for every $\emptyset \neq B \subseteq \mathbb{A}$)
$\hat{D}_B i \leftrightarrow \bigwedge_{b\in B} \hat{K}_b i$, (for all $i \in \mathsf{NOM}$ and all $\emptyset \neq B \subseteq \mathbb{A}$) DK
Rules for $\mathbf{K}_{\mathcal{PH}(-,D)}$:
All the rules for $\mathbf{K}_{\mathcal{PH}(-)}$
All the rules for $\mathbf{K}_{\mathcal{PH}(-)}$ involving K_a, with K_a replaced by D_B
(for every $\emptyset \neq B \subseteq \mathbb{A}$)

Figure 4.4: The Hilbert-style proof system for $\mathbf{K}_{\mathcal{PH}(-,D)}$.

In the standard way of adding distributed knowledge we first add distributed knowledge to the language $\mathcal{PH}(-)$. We will use the same approach as for the global modality: simply take the usual modal axioms and rules for the modalities D_B and add additional pure axioms. The proof system of the logic $\mathbf{K}_{\mathcal{PH}(-,D)}$ is given in Figure 4.4. The completeness proof follows from the general completeness in theorem 41 or 42, since the only new axiom DK is a pure formula. All that remains to be shown is that DK defines the right frame property. However, this is easily shown and stated as:

Lemma 47. *$\hat{D}_B i \leftrightarrow \bigwedge_{b\in B} \hat{K}_b i$ is valid on a frame $\langle W, (R_a)_{a\in\mathbb{A}}, (R_B)_{B\neq\emptyset, B\subseteq\mathbb{A}}\rangle$ if and only if $R_B = \bigcap_{b\in B} R_b$.*

[12] There is no axiom that for all frames $\langle W, R_1, R_2, R_3\rangle$ can force $R_1 = R_2 \cap R_3$, see for instance [126]. However the logic obtained by adding distributed knowledge, interpreted as intersection, to epistemic logic can be axiomatized, see for instance [57].

Theorem 48 (Completeness of $\mathbf{K}_{\mathcal{PH}(-,D)}$)**.** *Let Σ be a substitution-closed set of pure $\mathcal{PH}(-,D)$-formulas. Every set of $\mathcal{PH}(-,D)$-formulas that is $\mathbf{K}_{\mathcal{PH}(-,D)} + \Sigma$-consistent is satisfiable in a model whose underlying frame validates all the formulas in Σ.*

After adding distributed knowledge to $\mathbf{K}_{\mathcal{PH}(-)}$ we can now add it to $\mathbf{K}_{\mathcal{HPAL}(-)}$ using a reduction axiom. A reduction axiom for distributive knowledge similar to the one for K_a can be used, as already noted in [157]. The axiomatization of the Hybrid Public Announcement Logic, including distributed knowledge $\mathbf{K}_{\mathcal{HPAL}(-,D)}$ is shown in Figure 4.5. The soundness of the reduction axiom for D_B is guaranteed by the following lemma:

Lemma 49. *For all non-empty $B \subseteq \mathbb{A}$ and all $\mathcal{HPAL}(-,D)$-formulas φ and ψ, $[\varphi]D_B\psi$ is equivalent to $\varphi \to D_B[\varphi]\psi$.*

Proof. The proof is given by the following equivalences:

$$
\begin{array}{ll}
 & \mathcal{M}, w \models [\varphi]D_B\psi \\
\text{iff} & \mathcal{M}, w \models \varphi \Rightarrow \mathcal{M}|_\varphi, w \models D_B\psi \\
\text{iff} & \mathcal{M}, w \models \varphi \Rightarrow \Big(\forall v \in W|_\varphi \big[(w,v) \in \bigcap_{b\in B}(R_b \cap (W|_\varphi)^2) \Rightarrow \mathcal{M}|_\varphi, v \models \psi\big]\Big) \\
\text{iff} & \mathcal{M}, w \models \varphi \Rightarrow \Big(\forall v \in W|_\varphi \big[(w,v) \in (\bigcap_{b\in B} R_b) \cap (W|_\varphi)^2 \Rightarrow \mathcal{M}|_\varphi, v \models \psi\big]\Big) \\
\text{iff} & \mathcal{M}, w \models \varphi \Rightarrow \Big(\forall v \in W\big[\mathcal{M}, v \models \varphi \Rightarrow \big((w,v) \in (\bigcap_{b\in B} R_b) \cap (W|_\varphi)^2 \Rightarrow \mathcal{M}|_\varphi, v \models \psi)\big]\Big) \\
\text{iff} & \forall v \in W\Big[\mathcal{M}, w \models \varphi \Rightarrow \Big(\mathcal{M}, v \models \varphi \Rightarrow \big((w,v) \in (\bigcap_{b\in B} R_b) \cap (W|_\varphi)^2 \Rightarrow \mathcal{M}|_\varphi, v \models \psi\big)\Big)\Big] \\
\text{iff}^* & \forall v \in W\Big[\mathcal{M}, w \models \varphi \Rightarrow \Big(\mathcal{M}, v \models \varphi \Rightarrow \big((w,v) \in (\bigcap_{b\in B} R_b) \Rightarrow \mathcal{M}|_\varphi, v \models \psi\big)\Big)\Big] \\
\text{iff} & \forall v \in W\Big[\mathcal{M}, w \models \varphi \Rightarrow \Big((w,v) \in (\bigcap_{b\in B} R_b) \Rightarrow \big(\mathcal{M}, v \models \varphi \Rightarrow \mathcal{M}|_\varphi, v \models \psi\big)\Big)\Big] \\
\text{iff} & \forall v \in W\Big[\mathcal{M}, w \models \varphi \Rightarrow \big((w,v) \in (\bigcap_{b\in B} R_b) \Rightarrow \mathcal{M}, v \models [\varphi]\psi\big)\Big] \\
\text{iff} & \mathcal{M}, w \models \varphi \Rightarrow \forall v \in W\Big[(w,v) \in (\bigcap_{b\in B} R_b) \Rightarrow \mathcal{M}, v \models [\varphi]\psi\Big] \\
\text{iff} & \mathcal{M}, w \models \varphi \Rightarrow \mathcal{M}, w \models D_B[\varphi]\psi \\
\text{iff} & \mathcal{M}, w \models \varphi \to D_B[\varphi]\psi.
\end{array}
$$

In the equivalence "iff*" we have used the fact that $(w,v) \in (W|_\varphi)^2$ is equivalent to $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, v \models \varphi$. $\square$

With this lemma in place we have soundness of the logic $\mathbf{K}_{\mathcal{HPAL}(-,D)}$ and a completeness theorem for the logic, in the style of Theorem 46, can be proven in the same way as done in Section 4.3:

Axioms for $\mathbf{K}_{\mathcal{HPAL}(-,D)}$:

All axioms for $\mathbf{K}_{\mathcal{PH}(-,D)}$

All the relevant reduction axioms from Figure 4.2.

$[\varphi]\, D_B\psi \leftrightarrow (\varphi \to D_B[\varphi]\psi)$ [1] Announcement and distributed knowledge

Rules for $\mathbf{K}_{\mathcal{HPAL}(-,D)}$:

All rules for $\mathbf{K}_{\mathcal{PH}(-,D)}$

[1] Where B is a non-empty subset of $\mathbb{A}$

Figure 4.5: The Hilbert-style proof system for $\mathbf{K}_{\mathcal{HPAL}(@,\downarrow,E,D)}$.

Theorem 50 (Completeness for $\mathbf{K}_{\mathcal{HPAL}(-,D)}$)**.** *Let Σ be a set of pure $\mathcal{HPAL}(-,D)$-formulas. Every set of $\mathcal{HPAL}(-,D)$-formulas that is $\mathbf{K}_{\mathcal{HPAL}(-,D)}$+$\Sigma$-consistent is satisfiable in a model whose underlying frame validates all the formulas in Σ.*

4.4.2 Adding distributed knowledge directly

As the reader might have guessed, there is nothing to prevent adding distributed knowledge directly to $\mathbf{K}_{\mathcal{HPAL}(@,-)}$ using the pure formulas from the previous subsection. In other words an alternative proof system (however resulting in the same axioms and rules as the proof system of Figure 4.5) can be described as in Figure 4.6. Completeness of this proof system follows directly from Theorem 46. Thus we do not need to prove soundness of the reduction axiom for D_B, as it follows from the soundness of the reduction axiom for K_a.

The only thing that has to be verified in this way of adding distributed knowledge is that we get completeness with respect to the right class of frames. There is a little more subtleness to this than in the case of $\mathbf{K}_{\mathcal{PH}(-)}$. Theorem 46 only ensures that the axiom DK becomes valid in the underlying frame and not necessarily in all subframes of that frame. However if a frame satisfies that $R_B = \bigcap_{b\in B} R_b$, then all subframes also satisfy this property. Thus the meaning of D_B does not change after public announcement.

To see that adding axioms that are not valid in all subframes is a real problem, look at the modality $[a;b]$ defined by:

$$\mathcal{M}, w, g \models [a;b]\varphi \quad \text{iff} \quad \text{for all } v \in W;\ \text{if } (w,v) \in R_a;R_b \text{ then } \mathcal{M}, v, g \models \varphi,$$

where $R_a;R_b$ denotes the composition of the relations R_a and R_b defined by $R_a;R_b = \{(x,y) \mid \exists z : (x,z) \in R_a \wedge (z,y) \in R_b\}$. In classical Hybrid Logic

Axioms for $\mathbf{K}_{\mathcal{HPAL}(-,D)}$:
All the axioms for $\mathbf{K}_{\mathcal{HPAL}(-)}$
All the axioms of $\mathbf{K}_{\mathcal{HPAL}(-)}$ involving K_a, with K_a replaced by D_B (for every $\emptyset \neq B \subseteq \mathbb{A}$)
$\hat{D}_B i \leftrightarrow \bigwedge_{b \in B} \hat{K}_b i$, (for all $i \in \mathsf{NOM}$ and all $\emptyset \neq B \subseteq \mathbb{A}$) DK

Rules for $\mathbf{K}_{\mathcal{HPAL}(-,D)}$:
All the rules for $\mathbf{K}_{\mathcal{HPAL}(-)}$
All the rules of $\mathbf{K}_{\mathcal{HPAL}(-)}$ involving K_a, with K_a replaced by D_B (for every $\emptyset \neq B \subseteq \mathbb{A}$)

Figure 4.6: The alternative Hilbert-style proof system for $\mathbf{K}_{\mathcal{PH}(-,D)}$.

this is definable by the pure axiom $\langle a; b\rangle i \leftrightarrow \langle a\rangle\langle b\rangle i$. This axiom is easily seen to be valid exactly on the class of frames where $R_{a;b} = R_a; R_b$. However, just because $R_{a;b} = R_a; R_b$ holds on a frame, does not necessarily imply that it also holds on all subframes.[13] Thus in the scope of a public announcement operator $[\varphi]$ the modality $[a; b]$ will change its meaning in the sense that it does not quantify over the composition of the relations R_a and R_b in the submodel, but over the composition of the relations R_a and R_b in the original model. The problem lies in the fact that composition is not an operation that is preserved when going to submodels contrary to intersection. We will return to this issue in Section 4.4.4.

4.4.3 The definability of distributed knowledge using satisfaction operators and the downarrow binder

In the case of the logics $\mathbf{K}_{\mathcal{PH}(@,\downarrow,-)}$ (or $\mathbf{K}_{\mathcal{HPAL}(@,\downarrow,-)}$) it turns out that distributed knowledge is locally definable. The following proposition states this formally:

Proposition 51. *Let $B \subseteq \mathbb{A}$ contain at least 2 elements*[14]*, let $a \in B$, let φ be a $\mathcal{PH}(@, \downarrow, -)$-formula and let x and y be different state variables that do*

[13] Take for instance the frame $\langle W, R_a, R_b, R_{a;b}\rangle$, where $W = \{x, y, z\}$, $R_a = \{(x, y)\}$, $R_b = \{(y, z)\}$ and $R_{a;b} = \{(x, z)\}$. Clearly this frame satisfy that $R_{a;b} = R_a; R_b$, but in the subframe only containing the states x and z, we still have $(x, z) \in R_{a;b}$ although $(x, z) \notin R_a; R_b$.

[14] If B only contains b then clearly D_B is definable as K_b.

not occur in φ. Then for all models $\mathcal{M} = \langle W, (R_a)_{a\in\mathbb{A}}, V\rangle$, all assingments g in $\mathcal{M}$ and all $w \in W$:

$$\mathcal{M}, w, g \models D_B\varphi \quad \textit{iff} \quad \mathcal{M}, w, g \models {\downarrow}x.K_a {\downarrow}y.\big(@_x(\wedge_{b\in B\setminus\{a\}}\hat{K}_b y) \to \varphi\big).$$

Proof. The proof is given by the following equivalences, where g' is just like g except that $g'(x) = w$ and g'' is just like g' except that $g''(y) = v$ (thus g'' is just like g except that $g''(x) = w$ and $g''(y) = v$):

$$\begin{aligned}
& \mathcal{M}, w, g \models {\downarrow}x.K_a {\downarrow}y.\big(@_x(\wedge_{b\in B\setminus\{a\}}\hat{K}_b y) \to \varphi\big)\\
\text{iff}\quad & \mathcal{M}, w, g' \models K_a {\downarrow}y.\big(@_x(\wedge_{b\in B\setminus\{a\}}\hat{K}_b y) \to \varphi\big)\\
\text{iff}\quad & \forall v \in W\Big(wR_a v \;\Rightarrow\; \mathcal{M}, v, g' \models {\downarrow}y.\big(@_x(\wedge_{b\in B\setminus\{a\}}\hat{K}_b y) \to \varphi\big)\Big)\\
\text{iff}\quad & \forall v \in W\Big(wR_a v \;\Rightarrow\; \mathcal{M}, v, g'' \models @_x(\wedge_{b\in B\setminus\{a\}}\hat{K}_b y) \to \varphi\Big)\\
\text{iff}\quad & \forall v \in W\Big(wR_a v \;\Rightarrow\; \big(\mathcal{M}, w, g'' \models \wedge_{b\in B\setminus\{a\}}\hat{K}_b y \;\Rightarrow\; \mathcal{M}, v, g'' \models \varphi\big)\Big)\\
\text{iff}\quad & \forall v \in W\Big(wR_a v \;\Rightarrow\; \big(\forall b \in B\setminus\{a\}\,\exists s \in W(wR_b s \text{ and } \mathcal{M}, s, g'' \models y) \;\Rightarrow\; \mathcal{M}, v, g'' \models \varphi\big)\Big)\\
\text{iff}\quad & \forall v \in W\Big(wR_a v \;\Rightarrow\; \big(\forall b \in B\setminus\{a\}\,\exists s \in W(wR_b s \text{ and } s = v) \;\Rightarrow\; \mathcal{M}, v, g'' \models \varphi\big)\Big)\\
\text{iff}\quad & \forall v \in W\big(\forall b \in B(wR_b v) \;\Rightarrow\; \mathcal{M}, v, g'' \models \varphi\big)\\
\text{iff}^*\quad & \forall v \in W\big(\forall b \in B(wR_b v) \;\Rightarrow\; \mathcal{M}, v, g \models \varphi\big)\\
\text{iff}\quad & \mathcal{M}, w, g \models D_B\varphi,
\end{aligned}$$

where we in "iff*" have used that x and y do not occur in φ. □

Thus when adding distributed knowledge to the logics $\mathbf{K}_{\mathcal{HPAL}(@,\downarrow,-)}$ we can simply take the formula $D_B\varphi$ to be an abbreviation for the formula ${\downarrow}x.K_a {\downarrow}y.\big(@_x(\wedge_{b\in B\setminus\{a\}}\hat{K}_b y) \to \varphi\big)$. Furthermore, as a corollary, adding distributed knowledge does not add to the expressive power of $\mathbf{K}_{\mathcal{PH}(@,\downarrow,-)}$ or $\mathbf{K}_{\mathcal{HPAL}(@,\downarrow,-)}$:

Corollary 52. *The logics $K_{\mathcal{PH}(@,\downarrow,-)}$ ($K_{\mathcal{HPAL}(@,\downarrow,-)}$) and $K_{\mathcal{PH}(@,\downarrow,D,-)}$ ($K_{\mathcal{HPAL}(@,\downarrow,D,-)}$) are equally expressive.*

4.4.4 A general way of adding modalities to public announcement logic

In the paper [105] Barteld Kooi gives a general framework for showing completeness and expressiveness results for logics using already given reduction axioms. However, what seems as a next natural step, which is not considered in [105], is the actual question of how to find reduction axioms for a given modal operator relative to the public announcement operator. In this section we will take a first step towards answering this question by characterizing a class of modalities that have particularly simple reduction axioms. As mentioned in the example with composition in Section 4.4.2 this has to do with whether or not an operation is preserved when moving to submodels.

Note that the axioms for $@_i$ and E look alike and the axioms for K_a and D_B look alike. The difference between these two cases is alone due to the fact that $@_i$ and E are existential modalities, whereas K_a and D_B are universal modalities. Allowing for dual operators, we could write the four reduction axioms as one, namely

$$[\varphi]\Box\psi \quad \leftrightarrow \quad (\varphi \to \Box[\varphi]\psi), \tag{4.2}$$

where $\Box$ is one of K_a, $\overline{@}_i$, A or D_B. Equivalently we could use the axiom

$$[\varphi]\Diamond\psi \quad \leftrightarrow \quad (\varphi \to \Diamond(\varphi \wedge [\varphi]\psi)), \tag{4.3}$$

where $\Diamond$ is one of $\hat{K}_a$, $@_i$, E or $\hat{D}_B$. In the proof of soundness of the reduction axiom for distributed knowledge (Lemma 49) the only property of the semantics of D_B we used was the fact that $\bigcap_{b\in B}(R_b \cap (W|_\varphi)^2) = (\bigcap_{b\in B} R_b) \cap (W|_\varphi)^2$. The soundness of the reduction axioms for K_a, $@_i$ and E can be viewed as consequences of the same property. To show that this property always guarantees reduction axioms of the above form, we need to specify a general framework. Given a model $\mathcal{M} = \langle W, (R_a)_{a\in\mathbb{A}}, V\rangle$ when we speak of "a binary relation on $\mathcal{M}$" we simply mean a binary relation on W.

Definition 53. *A n-ary* model-relation-operation *is an operation that to any model $\mathcal{M}$ and n binary relations on $\mathcal{M}$ assigns a binary relation on $\mathcal{M}$.*

An example of such a model-relation-operation is intersection as used in the semantics for distributed knowledge. A n-ary model-relation-operation $\mathbf{Int}_n$ can be defined by $\mathbf{Int}_n(\mathcal{M}, R_1, ..., R_n) = \cap_{i=1...n} R_i$. Let a non-empty $B \subseteq \mathbb{A}$ be given and assume that B has n elements $b_1, ..., b_n$. Then for any model $\mathcal{M} = \langle W, (R_a)_{a\in\mathbb{A}}, V\rangle$, $\cap_{b\in B} R_b = \mathbf{Int}_n(\mathcal{M}, R_{b_1}, ..., R_{b_n})$. Thus the semantics of D_B can alternatively be specified as:

$$\mathcal{M}, w, g \models D_B\varphi \quad \text{iff} \quad \text{for all } v \in W: \text{ if } (w, v) \in \mathbf{Int}_n(\mathcal{M}, R_{b_1}, ..., R_{b_n}) \text{ then } \mathcal{M}, v, g \models \varphi.$$

Fixing a nominal i, a 0-ary model-relation-operation $\mathbf{Nom}_i$ can be defined by $\mathbf{Nom}_i(\mathcal{M}) = W \times \{V(i)\}$, where $W \times \{V(i)\}$ is the relation consisting of all pairs $(w, V(i))$ for $w \in W$ (if $V(i) = \emptyset$ then also $W \times \{V(i)\} = \emptyset$). The semantics of $\overline{@}_i\varphi$ can then be reformulated as:

$$\mathcal{M}, w, g \models \overline{@}_i\varphi \quad \text{iff} \quad \text{for all } v \in W: \text{ if } (w, v) \in \mathbf{Nom}_i(\mathcal{M}) \text{ then } \mathcal{M}, v, g \models \varphi.$$

Similar things can be done for the semantics of E and K_a by defining a 0-ary model-relation-operation **Glo** by $\mathbf{Glo}(\mathcal{M}) = W \times W$ and a unary model-relation-operation $\mathbf{Id}_1$ by $\mathbf{Id}_1(\mathcal{M}, R) = R$. Further examples of model-relation-operations are the PDL constructors union ($\cup$), composition (;) and transitive closure (*).

Definition 54. *An n-ary model-relation-operation* $\mathbf{F}$ respects intersection *if for all models* $\mathcal{M}$*, all relations* $R_1, ..., R_n$ *on* $\mathcal{M}$ *and any* $C \subseteq W \times W$*:*

$$\mathbf{F}(\mathcal{M}, R_1 \cap C, ..., R_n \cap C) = \mathbf{F}(\mathcal{M}, R_1, ..., R_n) \cap C.$$

Note, that all of the model-relation-operations $\mathbf{Int}_n$, $\mathbf{Nom}_i$, **Glo** and **Id** respect intersection (this is easy to see). This is also the case for the PDL constructor union, but not for the composition and transitive closure.[15]

The property of respecting intersection is the right property to ensure simple reduction axioms. However, before we can state this we need to add modalities based on model-relation-operations to the syntax of the language. To do this we assume a set of function symbols FSYM (all $F \in \mathsf{FSYM}$ is assumed to have a finite arity) which we use to refer to the model-relation-operations. Since these operations also take relations as argument, we need something to fill in as arguments in the function symbols and for this we use the set of agents $\mathbb{A}$ already given. Now to the syntax of our languages $\mathcal{PH}(-)$ and $\mathcal{HPAL}(-)$, we add a new modal operator $[F(a_1, ..., a_n)]$ for each n-ary function symbol $F \in \mathsf{FSYM}$ and each $a_1, ..., a_n \in \mathbb{A}$.

To give semantics for the new modalities we need to interpret the function symbols. For this we assume a *relation-interpretation* $\mathcal{I}$ that assigns an n-ary model-relation-operation $\mathcal{I}(F)$ to each $F \in \mathsf{FSYM}$ of arity n. With this fixed relation-interpretation $\mathcal{I}$ we can define the semantics of the new modalities by:

$$\begin{array}{ll} & \mathcal{M}, w, g \models [F(a_1, ..., a_n)]\varphi \\ \text{iff} & \text{for all } v \in W: \text{ if } (w, v) \in \mathcal{I}(F)\big(\mathcal{M}, R_{a_1}, ..., R_{a_n}\big) \text{ then } \mathcal{M}, v, g \models \varphi. \end{array} \tag{4.4}$$

Modalities defined using model-relation-operations that respect intersection have very simple reduction axioms in the form of (4.2) and (4.3). However, one cannot just add these reduction axioms to get a sound and complete

[15] In the case of the composition constructor take for instance $W = \{x, y, z\}$, $R_a = \{(x, y)\}$ and $R_b = \{(y, z)\}$. Then $R_a; R_b = \{(x, z)\}$ and thus $(R_a; R_b) \cap \{x, z\}^2 = \{(x, z)\}$. But $R_a \cap \{x, z\}^2 = \emptyset$ and $R_b \cap \{x, z\}^2 = \emptyset$, so $(R_a \cap \{x, z\}^2); (R_b \cap \{x, z\}^2) = \emptyset$.

logic, there has to be a sound and complete axiom system for the underlying logic as well (in some cases, such as distributed knowledge, this is easy since the modality is definable by pure formulas). But with this in mind our considerations can be summarized in the following proposition:

Proposition 55. *Let $F \in \mathsf{FSYM}$ be of arity n and such that $\mathcal{I}(F)$ respects intersection and let $a_1, ..., a_n \in \mathbb{A}$. Assume furthermore that there is a sound and complete axiom system for the logic $\mathbf{K}_{\mathcal{PH}(-,[F(a_1,...,a_n)])}$, then a sound and complete axiom system for the logic $\mathbf{K}_{\mathcal{HPAL}(-,[F(a_1,...,a_n)])}$ can be obtained by adding the reduction axiom*

$$[\varphi][F(a_1, ..., a_n)]\psi \;\leftrightarrow\; (\varphi \to [F(a_1, ..., a_n)][\varphi]\psi)$$

(together with the other relevant reduction axioms) to the axiom system of $\mathbf{K}_{\mathcal{PH}(-,[F(a_1,...,a_n)])}$. Furthermore the logic $\mathbf{K}_{\mathcal{HPAL}(-,[F(a_1,...,a_n)])}$ is no more expressive than $\mathbf{K}_{\mathcal{PH}(-,[F(a_1,...,a_n)])}$.

Proof. The proof is similar to the proof of Theorem 50 based on a rewriting of the proof of Lemma 49 using the assumption that $\mathcal{I}(F)$ respects intersection. □

This proposition gives a uniform way of adding $@_i$, E and D_B to the logic $\mathbf{K}_{\mathcal{HPAL}(-)}$. Furthermore since the PDL operator "∪" also respects intersection, the epistemic modality E_B reading "everybody amongst B knows that..." can also be added with a reduction axiom of the form $[\varphi]E_B\psi \leftrightarrow (\varphi \to E_B[\varphi]\psi)$.[16] Since the operator of composition does not respect intersection, we do not obtain a reduction axiom in the style of the ones for D_B or E_B.

We have presented the proposition as an extension of $\mathbf{K}_{\mathcal{HPAL}(-)}$, however, it is clear that it works for any extension of just classical PAL (without common knowledge). Another important remark is that we have presented the proposition in the setting of the basic logic $\mathbf{K}$, and it cannot just be extended to arbitrary extensions of $\mathbf{K}$. If we require that the relations $(R_a)_{a\in\mathbb{A}}$ of our models satisfy a certain property (like reflexivity or transitivity etc.), we cannot always be sure that the restricted relations $R_a \cap (W|_\varphi)^2$ also satisfy this property. Thus if the property is not preserved when taking intersection the proposition does not apply. Nevertheless, if it is preserved we can extend the result beyond $\mathbf{K}$. One example is assuming that all the relations R_a for $a \in \mathbb{A}$

[16] Note, however, that since $E_B\varphi$ is directly definable as $\bigwedge_{b\in B} K_b\varphi$, it is not necessary to have the modality explicit in the language as in the case of distributed knowledge in the logic $\mathbf{K}_{\mathcal{PH}(@,\downarrow,-)}$.

are equivalence relations, which is normally done in Epistemic Logic. Here there is no problem since restricting an equivalence relation R_a to $R_a \cap (W|_\varphi)^2$ gives rise to an equivalence relation. Finally, note that the proposition only provides sufficient and not necessary conditions for the existence of reduction axioms. Finding necessary conditions is left for further research.

4.4.5 A note on common knowledge

We now turn to common knowledge. For a non-empty subset $B \subseteq \mathbb{A}$, the common knowledge operator C_B is added to the language, with the reading of $C_B\varphi$ as "it is common knowledge among the agents in B that φ". C_B has the following semantics:

$$\mathcal{M}, w, g \models C_B\varphi \quad \text{iff} \quad \text{for all } v \in W;\ \text{if } (w,v) \in \left(\bigcup_{b \in B} R_b\right)^* \text{ then } \mathcal{M}, v, g \models \varphi,$$

where R^* denotes the reflexive transitive closure of the relation R.

Problems arise when one wants to combine public announcement logic with common knowledge in the sense that we cannot prove completeness using reduction axioms anymore. Reduction axioms for common knowledge simply do not exist. One solution is to generalize the notion of common knowledge to what is called relativized common knowledge. Relativized common knowledge is exactly the notion needed to get completeness via reduction axioms for public announcement, see [157]. We will not take on the enterprise of adding common knowledge or relativized common knowledge to $\mathbf{K}_{\mathcal{PH}(-)}$ or $\mathbf{K}_{\mathcal{HPAL}(-)}$. We simply mention common knowledge because the concept of respecting intersection makes it clear why reduction axioms such as the ones for K_a, E and D_B do not work for common knowledge. Common knowledge corresponds to the PDL operator of transitive closure, which does not respect intersection and cannot otherwise be defined.

4.5 Conclusion and further work

In this paper it has been shown that nominals, satisfaction operators, the downarrow binder, the global modality, and distributed knowledge can be added to the Public Announcement Logic. Furthermore general completeness results for extensions with pure formulas, a well celebrated result in Hybrid Logic, also transfer to the case of Hybrid Public Announcement Logic. Properties of both Hybrid Logic and Public Announcement Logic are thus preserved in the combination. The completeness is shown using reduction axioms as in classical Public Announcement Logic. Hence the public announcement operator does not increase the expressive power when added to Hybrid Logic. Using the terminology of [154], classical Hybrid Logic is not closed under relativization because nominals might lose their references in submodels, but relaxing Hybrid Logic to a logic with only partially denoting nominals, Hybrid Logic does become closed under relativization. Thus, the fact that the $[\varphi]$-operator does not add expressivity is preserved in hybrid extensions of the basic multi-modal logic.

That the nice properties of Hybrid Logic are preserved in the combination with Public Announcement Logic adds significantly to the proof theory of Public Announcement Logic. We have demonstrated this by adding distributed knowledge via pure formulas. It was also shown that distributed knowledge could actually be defined using satisfaction operators and the downarrow binder. That Hybrid Logic has much to offer the proof theory of Public Announcement Logic is also demonstrated by the tableau system developed in [80], but surely there is still much more that Hybrid Logic can offer to the proof theory of Public Announcement Logic. This is left for future research. Finally a sufficient requirement for the existence of reduction axioms in a general setting has been discussed. This naturally leads to the question of whether there is a semantic requirement to put on the operations of section 4.4 that exactly characterizes the operations that allow reduction axioms. We leave this as further work as well.

Another line of further research is to add common knowledge to the Hybrid Public Announcement Logic. However, as mentioned, this might not allow completeness via reduction axioms. Besides adding common knowledge there is also the question of extending the logic from Public Announcement Logic to full Dynamic Epistemic Logic. The problem here is that in full Dynamic Epistemic Logic there are epistemic actions that can expand a state into several states, and thus it is not clear anymore what nominals should denote.

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4.6 Appendix: Alternative semantics for the public announcement operator

In this section we discuss the relationship between the standard semantics and an alternative semantics for the public announcement operator.

Let us first fix the terminology. The semantics already introduced will be referred to as *the standard semantics*. By *the alternative semantics* we refer to the semantics of classical hybrid logic together with the following semantics for the public announcement operator $[\varphi]$:

$$\mathcal{M}, w \models [\varphi]\psi \quad \iff \quad \mathcal{M}, w \models \varphi \text{ implies that } \mathcal{M}|_\varphi, w \models \psi,$$

where $\mathcal{M}|_\varphi$ is the model $\mathcal{M} = \langle W, (R_a|_\varphi)_{a\in\mathbb{A}}, V\rangle$, where

$$R_a|_\varphi = R_a \cap (\{w \in W \mid \mathcal{M}, w \models \varphi\} \times \{w \in W \mid \mathcal{M}, w \models \varphi\}).$$

For the global modality it is easy to see that the logic obtained with the alternative semantics differs from the standard one. In the standard semantics the formula $[p]Ap$ is valid, but with the alternative semantics it is no longer valid. In a similar way we have that $[p]E\neg p$ becomes satisfiable. So after updating with the fact p there is still a state where p is false, which seems contra intuitive. In general publicly announcing a formula involving higher order knowledge might lead to it becoming false, but p is a propositional fact about the worlds which normally are assumed to be unchangeable by just announcements. This is also so in classical PAL. Thus when including the global modality it is more reasonable to use the standard semantics for the public announcement operator.

With the alternative semantics the satisfaction operator also get strange properties. Before an announcement it might be the case that $@_i p \wedge @_i \neg K_a p$, i.e. agent a does not know p at the state i. However, after announcing that the actual state is not i (i.e. a public announcement of $\neg i$) it becomes true that $@_i K_a p$ in the alternative semantics. This is essentially due to the following validity in the alternative semantics: $[\neg i](@_i\varphi \rightarrow @_i K_a \varphi)$. Thus information about which state is not the case gives complete information about the world at that state to every agent. Note that in the standard semantics announcing $\neg i$ simply makes every formula of the form $@_i\varphi$ false, which may not be a completely pleasing solution, but still the best one to the authors opinion.

Comparing the two logics related to the two semantics, note that $[p]Ap$ is valid in the standard semantics but not in the alternative one. On the other hand $Ep \rightarrow [\varphi]Ep$ is valid in the alternative semantics, but not in the standard one. Thus the two logics are simply different; none of them are contained in the other. Furthermore, and also important, it does not seem entirely clear how to derive reduction axioms for the logic of the alternative semantics.

Chapter 5

Terminating tableaux for dynamic epistemic logics

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Abstract: Throughout the last decade, there has been an increased interest in various forms of dynamic epistemic logics to model the flow of information and the effect this flow has on knowledge in multi-agent systems. This enterprise, however, has mostly been applicationally and semantically driven. This results in a limited amount of proof theory for dynamic epistemic logics.

In this paper, we try to compensate for a part of this by presenting terminating tableau systems for full dynamic epistemic logic with action models and for a hybrid public announcement logic (both without common knowledge). The tableau systems are extensions of already existing tableau systems, in addition to which we have used the reduction axioms of dynamic epistemic logic to define rules for the dynamic part of the logics. Termination is shown using methods introduced by Braüner, Bolander, and Blackburn.

Keywords: Dynamic epistemic logic, public announcement logic, terminating tableau systems, decision procedures, hybrid logic, reduction axioms.

5.1 Introduction

Classic epistemic logic has played an important role in both philosophy and computer science. However, recent years have witnessed the importance of also looking at the dynamics of knowledge, i.e. how knowledge of different agents can change due to the development of a system. There are two ways of adding dynamics to epistemic logic. One can either combine it with a temporal logic or combine it with some dynamic logic of actions. The latter approach has become increasingly common and has resulted in what is now called Dynamic Epistemic Logic (DEL), which includes operators for so called *epistemic actions* (*cf.* [163]). The interest in DEL has mostly been related to applications, and has mainly been semantically driven. Thus, only very few attempts to develop a rich proof theory for DEL beyond standard Hilbert style systems have been performed. This work attempts to make up for some of this by discussing terminating tableau systems for different kinds of dynamic epistemic logics.

The simplest form of dynamics one can add to classical epistemic logic is a public announcement operator. The language is extended with formulas of the type $[\varphi]\psi$, which are read as "after public announcement of φ, ψ holds". At the semantic level, the operator $[\varphi]$ corresponds to moving to the submodel consisting only of states where φ is true (thus we are exclusively concerned with truthful public announcements here). This simple extension, called Public Announcement Logic (PAL), is, nevertheless, quite useful as shown by the many applications presented in [163]. Having left out common knowledge, operators[1], this logic is fairly simple and a few tableau systems do already exist, see [14] and [49]. The approach in this paper is different from these in the sense that we try to avoid constructing new complicated and tailor made tableau systems by instead using the existing systems. This is possible due to reduction axioms. Reduction axioms have, from the beginning of DEL's short history, played an important role in showing completeness and expressiveness results. It was proved that Public Announcement Logic is no more expressive than the underlying epistemic logic. Using reduction axioms, it is possible to translate a public announcement formula into an equivalent one without any public announcement operators.

There are a lot of other possible epistemic actions, moving beyond bare public announcements: announcements to subgroups, private communications,

[1] In the rest of this paper, at least until the conclusion, we will disregard common knowledge.

secret announcements and more. The insight of Baltag, Moss and Solecki (in [15]) is that all these epistemic actions, considered as action modalities, can also be represented by a form of Kripke models. Using a general product operation on Kripke models, they can be given a semantic. More surprisingly, it was shown that also formulas with these more complicated action modalities can be reduced to basic epistemic formulas without any action modalities. This, of course, required more advanced reduction axioms than in the case of public announcements.

When one wants to prove a validity of DEL, one can simply translate the validity into an epistemic formula without action modalities and then use the existing tableau (or other) systems. However, the translation might result in an exponential increase in the size of the formula. As is shown in [115], this exponential increase cannot be avoided for public announcements. This fact provides another motivation for using DEL, since it offers us the opportunity to express things much more compactly than in classical epistemic logic. It is also shown in [115] that the complexity of validity checking for PAL is no higher than for the underlying epistemic logic. Thus, the method of first translating and then using known proof methods for classical epistemic logic may be unfeasible. This justifies the direct tableau systems for PAL given in [14] and [49]. In DELs with arbitrary epistemic actions, the matter becomes much more complicated. Here, the challenge is how to represent the action modalities. Since PAL is part of DEL, [115] also shows that the blowup of the translation may be exponential in the general DEL case. However, it is currently unknown to the author exactly what the complexity of deciding DEL validities is.[2] When adding the global modality to the underlying epistemic logic, the complexity of this will already be exponential [144]. In this case, the exponential increase caused by the translation therefore does not destroy the worst-case complexity.

The work in this paper is based on the idea of using reduction axioms as rules to make the translation on the fly in the tableaux. In practice, this is more efficient than performing the whole translation at the beginning[3], but in

[2]When dealing with arbitrary formulas of DEL, the question is how to measure the size of the action modalities. On the one hand, an action modality could be counted as one symbol, but when deciding validity, the finer structure of the action modality is needed. Thus, this may result in a high complexity for validity checking in this size of the formula. On the other hand, using another size-measure of action modalities, it may become possible to decide validity in lower complexity in that size.

[3]In worst case scenario doing the translation on the fly may not be more efficient. But there seems to be at least two cases where translation on the fly will speed up the process

the case of public announcement, it might not be as fast as the methods of [14] and [49]. However, their tableau systems only work for public announcement logic, while the method presented here further works for full dynamic epistemic logic and for a hybrid version of public announcement logic. Our tableaux in this paper are kept terminating using the methods of Braüner, Bolander, and Blackburn ([38], [39], and [37]). The presentation here will be based on the approach in [37]. For basic modal logic, they show termination by noticing that the maximal formula complexity drops as new prefixes are introduced, which makes infinite tableaux impossible. In this paper we show that, essentially, this argument can be adapted in the setting where reduction axioms are used as extra tableau rules.

The paper is structured as follows: first we introduce public announcement logic, a hybrid public announcement logic, and full dynamic epistemic logic (Section 5.2). Then, we present a terminating tableau system for full dynamic epistemic logic in Section 5.3. Following this, we demonstrate how the approach can also be used to create a terminating tableau system for the hybrid public announcement logic. Finally, we present some concluding remarks and discuss further research.

5.2 Dynamic epistemic logic

We will first present the formal definitions of public announcement logic. Public announcements are added to the normal modal logic **K**, but it can easily be extended to the case of **S5**, which is often used for modeling knowledge. We will leave out common knowledge. First, we assume a finite set of agents $\mathbb{A}$ and a countable infinite set of propositional variables PROP. Using the terminology of [163] the language of the Public Announcement Logic will be denoted by $\mathcal{L}_{K[]}$, and is given by the following syntax:

$$\varphi \ ::= \ p \ | \ \neg\varphi \ | \ \varphi \wedge \varphi \ | \ K_a\varphi \ | \ [\varphi]\varphi,$$

where $p \in \mathsf{PROP}$ and $a \in \mathbb{A}$. The connectives $\vee$, $\rightarrow$ and $\leftrightarrow$ are defined from $\neg$ and $\wedge$ in the usual way, and the dual operators $\neg K_a \neg$ and $\neg[\varphi]\neg$ are abbreviated by $\hat{K}_a$ and $\langle\varphi\rangle$. The interpretation of $K_a\varphi$ is that "agent a knows

of deciding a formula. The first case is where only few steps of translation are needed to detect an inconsistency, as for instance in the formula $[\neg[\neg(q \wedge r)]K_a(p \rightarrow q)](p \vee r) \wedge \neg((\neg[\neg(q \wedge r)]K_a(p \rightarrow q)) \rightarrow (p \vee r))$. The other case is where the need for a translation may only occur at the very end of the tableau construction process, as for instance in the formula $K_a[p]p \wedge \neg K_a \neg[q]\neg q \wedge K_a([q]p \wedge [p]q)$.

that φ" and of $[\varphi]\psi$ that "after (truthful) public announcement of φ, ψ is the case". These interpretations are captured by the following formal semantic: A Kripke frame (or just a frame) is a pair $\mathcal{F} = \langle W, (R_a)_{a\in\mathbb{A}}\rangle$ consisting of a non-empty set W of states (or possible worlds) and for each $a \in \mathbb{A}$ a binary relation R_a on W (i.e. $R_a \subseteq W \times W$). A model $\mathcal{M}$ is a pair consisting of a frame $\mathcal{F}$ and a valuation V that assigns a set of states in W to every propositional variable of PROP (i.e. $V : \mathsf{PROP} \to \mathcal{P}(W)$). Given a formula φ of $\mathcal{L}_{K[]}$, a model $\mathcal{M} = \langle w, (R_a)_{a\in\mathbb{A}}, V\rangle$, and a state $w \in W$, the truth of φ at w, notation $\mathcal{M}, w \models \varphi$, is defined as standard in modal logic, taking K_a to be the box modality corresponding to R_a. In addition we add the following clause for $[\varphi]\psi$:

$$\mathcal{M}, w \models [\varphi]\psi \quad \textit{iff} \quad \mathcal{M}, w \models \varphi \textit{ implies that } \mathcal{M}|_\varphi, w \models \psi,$$

where the model $\mathcal{M}|_\varphi = \langle W|_\varphi, R|_\varphi, V|_\varphi\rangle$ is defined by:

$$\begin{array}{lcl} W|_\varphi & = & \{v \in W \mid \mathcal{M}, v \models \varphi\} \\ R_a|_\varphi & = & R_a \cap (W|_\varphi \times W|_\varphi) \\ V|_\varphi(p) & = & V(p) \cap W|_\varphi. \end{array}$$

We write $\mathcal{M} \models \varphi$ if $\mathcal{M}, w \models \varphi$ for all $w \in W$ and we say that φ is valid if $\mathcal{M} \models \varphi$ for all models $\mathcal{M}$. The logic of this semantic will be denoted by **PA** and be call Public Announcement Logic. It is not hard to prove the following validities in this logic:

$$[\varphi]\,p \quad\leftrightarrow\quad (\varphi \to p) \tag{5.1}$$

$$[\varphi]\,\neg\psi \quad\leftrightarrow\quad (\varphi \to \neg[\varphi]\psi) \tag{5.2}$$

$$[\varphi]\,(\psi \wedge \chi) \quad\leftrightarrow\quad ([\varphi]\psi \wedge [\varphi]\chi) \tag{5.3}$$

$$[\varphi]\,K_a\psi \quad\leftrightarrow\quad (\varphi \to K_a[\varphi]\psi) \tag{5.4}$$

$$[\varphi]\,[\psi]\chi \quad\leftrightarrow\quad [\varphi \wedge [\varphi]\psi]\chi. \tag{5.5}$$

These are the reduction axioms for public announcement [163]. Adding these axioms together with necessitation of $[\varphi]$ to a Hilbert style proof system for multi modal **K** will result in a sound and complete proof system for **PA** (for details see [163]).

Note that the complexity of the formula occurring within the scope of the public announcement operator is greater on the left than on the right side of "$\leftrightarrow$" in these reduction axioms. This can be used to define a translation $T : \mathcal{L}_{K[]} \to \mathcal{L}_K$, where $\mathcal{L}_K$ is the standard multi-modal language. The translation "commutes" with all logic operators beside the public announcement operator (e.g. $T(\neg\varphi) = \neg T(\varphi)$). In the case when the translation encounter a $[\varphi]$ operator it uses the reduction axioms to decrease the complexity of the formula within the scope of the operator, e.g. $T([\varphi](\neg\psi)) = T(\varphi) \to \neg T([\varphi]\psi)$. This translation can be shown to be a truth preserving translation of **PA** into multi modal **K**, which shows that adding the public announcement operator does not increase the expressiveness of the language.[4] This recursive translation is not recursive in the normal way, since going from left to right (of $\leftrightarrow$) in the reduction axioms (5.1)–(5.5) does not reduce the standard formula complexity. Therefore, to prove the correctness of the translation, a new complexity measure on the formulas is needed. One possible measure is (taken from [163]):

Definition 56. *Define a complexity measure $c : \mathcal{L}_{K[]} \to \mathbb{N}$ by the inductive clauses:*

$$\begin{array}{lcl} c(p) & = & 1 \\ c(\neg\varphi) & = & 1 + c(\varphi) \\ c(\varphi \wedge \psi) & = & 1 + \max\{c(\varphi), c(\psi)\} \\ c(K_a\varphi) & = & 1 + c(\varphi) \\ c([\varphi]\psi) & = & (4 + c(\varphi)) \cdot c(\psi) \end{array}$$

What can be shown about this complexity measure is that it decreases when moving to subformulas and, furthermore, that the c complexities of the left hand sides of the reduction axioms (5.1)–(5.5) are higher than the c complexities of the right hand sides. This fact will be important when we consider the tableau system in the next section.

We will not present a tableau system for **PA**, but for a hybrid extension of this, namely the Hybrid Public Announcement Logic of [81] (Chapter 4). To obtain this new logic we will first extend the language. For this we fix a countable infinite set of nominals NOM disjoint from the propositional variables. The language of hybrid public announcement logic $\mathcal{L}_{HPA}$ is defined by:

$$\varphi \ ::= \ p \mid i \mid \neg\varphi \mid \varphi \wedge \varphi \mid K_a\varphi \mid [\varphi]\varphi \mid @_i\varphi \mid E\varphi,$$

[4] This also works when the underlying logic is **S5**, however, if one wants to models beliefs using the logic **KD45** a problem arise. The problem is that the frame properties defined by the axioms of **KD45** are not preserved under the operation of taking submodels. Thus one cannot get completeness with respect to the class of models where beliefs are always interpreted as **KD45**. In other words the given semantic for the public announcement operator can result in agents having inconsistent beliefs after a public announcement.

where $p \in \mathsf{PROP}$, $i \in \mathsf{NOM}$ and $a \in \mathbb{A}$.

The nominals will function as names for states. The formula $@_i\varphi$ states that φ is true at the state that i denotes and $E\varphi$ express that there is a state where φ is true. The semantics is specified somewhat different from what is standard in hybrid logic. The reason is that the semantic of the public announcement operator takes us to submodels where states denoted by nominals may disappear. To deal with this, we extend the class of models such that the valuation assigns *at most one* state instead of *exactly one* state to each nominal (for more on these issues, see [81]). The definition of a model $\mathcal{M} = \langle W, (R_a)_{a\in\mathbb{A}}, V\rangle$ is thus the same as for **PA**, but with the further requirement on the valuation $V : \mathsf{PROP} \cup \mathsf{NOM} \to \mathcal{P}(W)$ that $|V(i)| \leq 1$ for all $i \in \mathsf{NOM}$. For the part of the language that coincide with $\mathcal{L}_{K[]}$ the semantic clauses are the same as for **PA**. For the new part of the language we define:

$$\begin{array}{lll} \mathcal{M}, w \models i & \textit{iff} & w \in V(i) \\ \mathcal{M}, w \models @_i\varphi & \textit{iff} & \textit{there is a } v \in V(i) \textit{ such that } \mathcal{M}, v \models \varphi \\ \mathcal{M}, w \models E\varphi & \textit{iff} & \textit{there is a } v \in W \textit{ such that } \mathcal{M}, v \models \varphi. \end{array}$$

The logic, this semantic give rise to, will be called "Hybrid Public Announcement Logic" and will be denoted by **HPA**. The dual operators of E and $@_i$ will be denoted by A and $\overline{@}_i$. Note, that since nominals only partially denote states, $@_i$ is no longer its own dual. We still have the equivalences $@_i\varphi \equiv E(i \wedge \varphi)$ and $\overline{@}_i\varphi \equiv A(i \to \varphi)$ though, but now these might not be equivalent anymore.[5] Thus the satisfaction operator has been split into an existential quantifier $@$ and an universal one $\overline{@}$. The fact, that the nominal i denotes something in a model (i.e. $|V(i)| = 1$) can be expressed by the formula Ei.

Completeness with respect to a Hilbert style proof system can also be shown using reduction axioms, as the following from [81]:

$$[\varphi]\, i \quad\leftrightarrow\quad (\varphi \to i) \tag{5.6}$$

$$[\varphi]\, @_i\psi \quad\leftrightarrow\quad (\varphi \to @_i(\varphi \wedge [\varphi]\psi)) \tag{5.7}$$

$$[\varphi]\, E\psi \quad\leftrightarrow\quad (\varphi \to E(\varphi \wedge [\varphi]\psi)) \tag{5.8}$$

Since we extended the language of $\mathcal{L}_{K[]}$ we also need to extend the definition

[5] These equivalences also show that we do not need the $@_i$ operator in the language, since it is definable in terms of E and i. However to ease the adaption of the tableau system from [37] we keep $@_i$ in the language.

of the measure c. This done by adding the following clauses to Definition 56:

$$\begin{array}{rcl} c(i) & = & 1 \\ c(@_i\varphi) & = & 1 + c(\varphi) \\ c(E\varphi) & = & 1 + c(\varphi). \end{array}$$

With this complexity measure, the left hand sides of the new reduction axioms (5.6)–(5.8) still have higher c complexity than the right hand sides.

Public announcements are just one kind of epistemic action though. To deal with a larger amount of epistemic actions in a uniform way, the notion of action models was introduced by Baltag, Moss and Solecki [15]. The intuition behind epistemic action models is that the agents may be unsure about exactly which action takes place and that each action has a precondition that has to be satisfied for that action to take place. Epistemic actions can be represented by Kripke structures where each state is an event/action and instead of a complete valuation each event is assigned a formula of the language as a precondition.

We now turn to the formal details. An action model $\mathsf{M} = \langle \mathsf{S}, (\mathsf{Q_a})_{a\in\mathbb{A}}, \mathsf{pre}\rangle$ consists of a finite set of events S, accessibility relations $\mathsf{Q_a}$ on S for all agents $a \in \mathbb{A}$, and a precondition function $\mathsf{pre} : \mathsf{S} \to \mathcal{L}$ assigning a precondition to every event (for some logical language $\mathcal{L}$). The language of formulas, $\mathcal{L}_{K\otimes}$, and the action model language, $\mathcal{L}^{act}_{K\otimes}$, have to be defined at the same time using mutual recursion. The action model language $\mathcal{L}^{act}_{K\otimes}$ is defined by:

$$\alpha \ ::= \ (\mathsf{M}, \mathsf{s}),$$

where $\mathsf{M} = \langle \mathsf{S}, (\mathsf{Q_a})_{a\in\mathbb{A}}, \mathsf{pre}\rangle$ is an action model such that $\mathsf{s} \in \mathsf{S}$ and $\mathsf{pre} : \mathsf{S} \to \mathcal{L}_{K\otimes}$. At the same time the formula language $\mathcal{L}_{K\otimes}$ is defined by:

$$\varphi \ ::= \ p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K_a\varphi \mid [\alpha]\varphi,$$

where $p \in \mathsf{PROP}$, $a \in \mathbb{A}$, and $\alpha \in \mathcal{L}^{act}_{K\otimes}$.[6] The reading of the formula $[\mathsf{M}, \mathsf{s}]\varphi$ is "after the epistemic action (M, s), φ is the case". M represent the uncertainty among the agents about what event is taking place, and s is the event that actually takes place.

[6]By this definition we load the syntax of the language with heavy semantic machinery, however, since we only deal with finite action models it is possible to list and name them all. For more on this discussion see [163].

For general epistemic actions a little contemplation shows that they can actually result in an enlargement of a Kripke model. The way we reflect this in the semantic is by defining a product update between a Kripke model and an action model. For a Kripke model $\mathcal{M} = \langle W, (R_a)_{a\in\mathbb{A}}, V\rangle$ and an action model $\mathsf{M} = \langle \mathsf{S}, (\mathsf{Q_a})_{a\in\mathbb{A}}, \mathsf{pre}\rangle$ define the restricted product $\mathcal{M}\otimes\mathsf{M} = \langle W', (R'_a)_{a\in\mathbb{A}}, V'\rangle$ by:

$$\begin{array}{lcl} W' & = & \{(w,\mathsf{s}) \in \mathcal{M}\times\mathsf{M} \mid \mathcal{M}, w \models \mathsf{pre}(\mathsf{s})\} \\ R'_a\big((w,\mathsf{s}),(v,\mathsf{t})\big) & \textit{iff} & R_a(w,v) \ \ \textit{and} \ \ \mathsf{Q_a}(\mathsf{s},\mathsf{t}) \\ V'(p) & = & \{(w,\mathsf{s}) \in W' \mid w \in V(p)\}. \end{array}$$

We can now define the semantic of the action modality $[\mathsf{M},\mathsf{s}]$ as:

$$\mathcal{M}, w \models [\mathsf{M},\mathsf{s}]\varphi \quad \textit{iff} \quad \mathcal{M}, w \models \mathsf{pre}(\mathsf{s}) \ \textit{implies that} \ \mathcal{M}\otimes\mathsf{M}, (w,\mathsf{s}) \models \varphi.$$

The other logical operators have the normal semantic and validity is also defined in the standard way. This logic will be called Dynamic Epistemic Logic and be denoted by **AM**. Note that we have now left out the hybrid machinery since it is not obvious how exactly to combine it with action models.[7]

As in the case of public announcement, adding action modalities does not increase the expressive power of the language. Again this is shown by providing reduction axioms (see for instance [163]). The reduction axioms, which are now a little more complex, are:

$$[\mathsf{M},\mathsf{s}]\, p \quad \leftrightarrow \quad \big(\mathsf{pre}(\mathsf{s}) \to p\big) \tag{5.9}$$

$$[\mathsf{M},\mathsf{s}]\, \neg\varphi \quad \leftrightarrow \quad \big(\mathsf{pre}(\mathsf{s}) \to \neg[\mathsf{M},\mathsf{s}]\varphi\big) \tag{5.10}$$

$$[\mathsf{M},\mathsf{s}]\, (\varphi\wedge\psi) \quad \leftrightarrow \quad \big([\mathsf{M},\mathsf{s}]\varphi \wedge [\mathsf{M},\mathsf{s}]\psi\big) \tag{5.11}$$

$$[\mathsf{M},\mathsf{s}]\, K_a\varphi \quad \leftrightarrow \quad \big(\mathsf{pre}(\mathsf{s}) \to \bigwedge_{R_a(\mathsf{s},\mathsf{t})} K_a[\mathsf{M},\mathsf{t}]\varphi\big) \tag{5.12}$$

$$[\mathsf{M},\mathsf{s}]\, [\mathsf{M}',\mathsf{s}']\varphi \quad \leftrightarrow \quad [(\mathsf{M};\mathsf{M}'),(\mathsf{s},\mathsf{s}')]\varphi, \tag{5.13}$$

where, in the last formula ,the ";" operation is a semantic operation on action models. Given two action models, $\mathsf{M} = \langle \mathsf{S}, (\mathsf{Q_a})_{a\in\mathbb{A}}, \mathsf{pre}\rangle$ and $\mathsf{M}' =$

[7]The interpretation of nominals is none obvious when modalities capable of expanding states are present. Normally, nominals in hybrid logic are a special kind of propositional variables, which are true in *exactly one* state. However, when taking a product of an epistemic model with an action model, single states of the epistemic model can turn into several states in the resulting product model. Thus, if one keeps the original definition of the valuation for the product model, one breaks the requirement of nominals only being true in one state. On the other hand, there seems to be no obvious alternative definition of the valuation.

$\langle \mathsf{S}', (\mathsf{Q}'_\mathsf{a})_{a\in\mathbb{A}}, \mathsf{pre}'\rangle$, the composition $(\mathsf{M};\mathsf{M}') = \langle \mathsf{S}'', (\mathsf{Q}''_\mathsf{a})_{a\in\mathbb{A}}, \mathsf{pre}''\rangle$ is defined by:

$$\begin{array}{lcl} \mathsf{S}'' & = & \mathsf{S}\times\mathsf{S}' \\ \mathsf{Q}''_\mathsf{a}\big((\mathsf{s},\mathsf{s}'),(\mathsf{t},\mathsf{t}')\big) & \textit{iff} & \mathsf{Q}_\mathsf{a}(\mathsf{s},\mathsf{t}) \ \ \textit{and} \ \ \mathsf{Q}'_\mathsf{a}(\mathsf{s}',\mathsf{t}') \\ \mathsf{pre}''((\mathsf{s},\mathsf{s}')) & = & \langle \mathsf{M},\mathsf{s}\rangle\mathsf{pre}'(\mathsf{s}'). \end{array}$$

See [163] for the validity of the reduction axioms. As for **HPA** a new complexity measure is needed. A such, taken from [163], is:

Definition 57. *The complexity measure $d : \mathcal{L}_{K\otimes} \to \mathbb{N}$ is defined inductively by:*

$$\begin{array}{lcl} d(p) & = & 1 \\ d(\neg\varphi) & = & 1 + d(\varphi) \\ d(\varphi\wedge\psi) & = & 1 + \max\{d(\varphi), d(\psi)\} \\ d(K_a\varphi) & = & 1 + d(\varphi) \\ d([\mathsf{M},\mathsf{s}]\varphi) & = & \big(4 + d(\mathsf{M},\mathsf{s})\big)\cdot d(\varphi) \\ d(\mathsf{M},\mathsf{s}) & = & \max\{d(\mathsf{pre}(\mathsf{t})) \mid \mathsf{t}\in\mathsf{M}\}. \end{array}$$

As for public announcement it can be shown that this complexity measure decreases when moving from the left hand sides to the right hand sides (of $\leftrightarrow$) of the reduction axioms (5.9)–(5.13), as well as when moving to subformulas.

5.3 A tableaux system for AM

In this section we introduce a tableau system for **AM** built on an existing tableau system for multi modal **K**, where we add the reduction axioms as tableau rules. This is done without violating termination or completeness of the original system. Formally, we will take a tableau to be a finitely branching tree, where each node is labeled by a formula of our language. As basic tableau system for the underlying multi modal **K** logic we will use the one from [37], which is a standard one.[8] The tableau system is a prefixed tableau system, thus all formulas occurring on the tableaux have the form $\sigma\varphi$, where σ comes from some fixed countable infinite set of prefixes. The intuition behind the prefixes is that they represent states in a possible Kripke model. Thus the intuition behind $\sigma\varphi$ is that φ holds at σ. Additionally, we also have formulas of the form $\sigma R_a \tau$ on the tableaux representing that τ is accessible from σ by

[8] The tableau system of [37] is also shown in Figure 1.5 of Section 1.2.1. The only change we have made to that system is the addition of the rules ($[AM]$) and ($\neg[AM]$).

$$\frac{\sigma\neg\neg\varphi}{\sigma\varphi}\,(\neg\neg) \qquad \frac{\sigma\varphi\wedge\psi}{\begin{array}{c}\sigma\varphi\\ \sigma\psi\end{array}}\,(\wedge) \qquad \frac{\sigma\neg(\varphi\wedge\psi)}{\sigma\neg\varphi \mid \sigma\neg\psi}\,(\neg\wedge)$$

$$\frac{\sigma\neg K_a\varphi}{\begin{array}{c}\sigma R_a\tau\\ \tau\neg\varphi\end{array}}\,(\neg K_a)^1 \qquad \frac{\sigma K_a\varphi \quad \sigma R_a\tau}{\tau\varphi}\,(K_a)$$

$$\frac{\sigma[\mathsf{M},\mathsf{s}]\varphi}{\sigma t([\mathsf{M},\mathsf{s}]\varphi)}\,([AM]) \qquad \frac{\sigma\neg[\mathsf{M},\mathsf{s}]\varphi}{\sigma\neg t([\mathsf{M},\mathsf{s}]\varphi)}\,(\neg[AM])$$

[1] The prefix τ is new to the branch.

Figure 5.1: Tableau rules **AM**.

agent a. These will be called accessibility formulas. The rules of the tableau system applies to branches of tableaux and are presented in Figure 5.1.

In the rules $([AM])$ and $(\neg[AM])$, t is the operation that uses the reduction axioms to translate the formula $[\mathsf{M},\mathsf{s}]\varphi$ to a formula of less d-complexity. For instance $t([\mathsf{M},\mathsf{s}](\varphi\wedge\psi)) = [\mathsf{M},\mathsf{s}]\varphi \wedge [\mathsf{M},\mathsf{s}]\psi$.[9] Ignoring the accessibility formulas, the formula above the line in a rule will be called the *premise* and the formula(s) below the line the *conclusion(s)*. When constructing a tableau, we never add a formula to a branch if it already occurs on the branch, and we never apply the $(\neg K_a)$ rule twice to the same formula on a branch. If a branch contains both $\sigma\varphi$ and $\sigma\neg\varphi$ for some formula φ and some prefix σ, then the branch is called *closed*, otherwise *open*. A closed tableau is one in which all branches are closed. A tableau proof of a formula φ is a closed tableau with $\sigma\neg\varphi$ as the root formula.

5.3.1 Termination of the tableau system

Two important properties for ensuring termination in the work of [37] are; *all formulas occurring on the tableau are subformulas or negation of subformulas of the root formula*, and *every rule only generates something of less formula complexity.* These two properties are essential for ensuring finiteness of the tableaux. However, these properties fail for our tableau system because the rules $([AM])$ and $(\neg[AM])$ can generate formulas that are not subformulas of

[9] t is not to be confused with a full translation for the language $\mathcal{L}_{K\otimes}$ as discussed in Section 5.2. Here t only translate/reduces one level.

the premise and may have higher formula complexity. But using the notion of d-complexity and stretching the notion of a subformula we can retain finiteness. Before a new notion of subformula can be defined a lemma is needed. For an action model $\mathsf{M} = \langle \mathsf{S}, (\mathsf{Q_a})_{a\in\mathbb{A}}, \mathsf{pre}\rangle$, let $\mathcal{D}(\mathsf{M})$ denote the domain, i.e. $\mathcal{D}(\mathsf{M}) = \mathsf{S}$.

Lemma 58. *Let $\sigma_0\varphi_0$ be the root formula of a tableau $\mathcal{T}$ and assume that $[\mathsf{M},\mathsf{s}]\varphi$ occurs on $\mathcal{T}$. Then, there are an $n \leq d(\varphi_0)$ and action models $\mathsf{M}_1, ..., \mathsf{M}_n$ occurring in φ_0, such that $\mathcal{D}(\mathsf{M}) = \mathcal{D}(\mathsf{M}_1) \times ... \times \mathcal{D}(\mathsf{M}_n)$.*

Proof. The proof goes by induction on the construction of $\mathcal{T}$. The claim is obvious for $[\mathsf{M},\mathsf{s}]\varphi$ being φ_0. It is also obvious that when applying any rule, besides ($[AM]$) and ($\neg[AM]$) applied to formulas of the form $[\mathsf{M},\mathsf{s}][\mathsf{M}',\mathsf{s}']\psi$ and $\neg[\mathsf{M},\mathsf{s}][\mathsf{M}',\mathsf{s}']\psi$, the action modalities in the conclusions and the premises have the same domains (or the action modality have completely been removed). Now for the rules ($[AM]$) and ($\neg[AM]$) applied to two consecutive modalities $[\mathsf{M},\mathsf{s}]$ and $[\mathsf{M}',\mathsf{s}']$. Assume that the claim of the lemma is true for $[\mathsf{M},\mathsf{s}]$ and $[\mathsf{M}',\mathsf{s}']$. Then

$$\mathcal{D}\Big((\mathsf{M};\mathsf{M}'),(\mathsf{s},\mathsf{s}')\Big) = \mathcal{D}(\mathsf{M}_1) \times ... \times \mathcal{D}(\mathsf{M}_n) \times \mathcal{D}(\mathsf{M}'_1) \times ... \times \mathcal{D}(\mathsf{M}'_m), \tag{5.14}$$

where M_i and M'_j occur in φ_0 for all $1 \leq i \leq n$ and $1 \leq j \leq m$. Thus, the only thing that remains to be shown is that $n + m \leq d(\varphi_0)$. Note, that the complexity measure d is an upper bound for how deep the action modalities can be nested. Furthermore, for every number of nested action modalities, only one more "$\times\mathcal{D}(\mathsf{M}_i)$" can be added by the rules ($[AM]$) and ($\neg[AM]$) to (5.14), which also decrease the number of nested modalities by one. Thus $n + m$ must be less than $d(\varphi_0)$. $\square$

Definition 59. *A formula ψ is said to be a d-*subformula *of a formula φ if*

- *$d(\psi) \leq d(\varphi)$,*
- *Every propositional variable p that occurs in ψ also occurs in φ.*
- *If an action modality $[\mathsf{M},\mathsf{s}]$ occurs in ψ, then there are action models $(\mathsf{M}_1,\mathsf{s}_1), ..., (\mathsf{M}_\mathsf{n},\mathsf{s}_n)$ for which M_i occurs in φ for $1 \leq i \leq n \leq d(\varphi)$, and $\mathcal{D}(\mathsf{M}) = \mathcal{D}(\mathsf{M}_1) \times ... \times \mathcal{D}(\mathsf{M}_n)$,*

Note that if the action modality $[\mathsf{M},\mathsf{s}]$ occurs in a formula φ, then all the preconditions of M are also counted as occurring in φ and by definition of the

d-complexity we automatically have that $d(\mathsf{pre}(\mathsf{t})) < d(\varphi)$ for all $\mathsf{t} \in \mathcal{D}(\mathsf{M})$. Using the reduction axioms as rules result in a decrease in d-complexity, since

$$d([\mathsf{M}, \mathsf{s}]\varphi) > d(t([\mathsf{M}, \mathsf{s}]\varphi))$$

Furthermore the d-complexity decreases when moving to a strict subformula. Thus we get the following lemma and from it a subformula property.

Lemma 60. *For every tableau rule the d-complexity of the conclusion is strictly less than the d-complexity of the premise.*

Lemma 61 (d-subformula property)**.** *Let $\mathcal{T}$ be a tableau with $\sigma_0\varphi_0$ as root formula. Then for every prefixed formula $\sigma\varphi$ on $\mathcal{T}$, φ is a d-subformula of φ_0.*

Proof. Let $\mathcal{T}$ be a tableau with $\sigma_0\varphi_0$ as root formula. The proof goes by induction on the tableau construction. By Lemma 60 it follows that the d-complexity of any formula occurring on $\mathcal{T}$ is less than $d(\varphi_0)$. Moreover it is obvious that none of the rules can introduce propositional variables that do not already occur in the root formula. The only rules that can introduce new action modalities are the ($[AM]$) and ($\neg[AM]$) rule applied to formulas $[\mathsf{M}, \mathsf{s}]K_a\varphi$, $\neg[\mathsf{M}, \mathsf{s}]K_a\varphi$, $[\mathsf{M}, \mathsf{s}][\mathsf{M}', \mathsf{s}']\varphi$, and $\neg[\mathsf{M}, \mathsf{s}][\mathsf{M}', \mathsf{s}']\varphi$. For the first two cases, a new action modality of the form $[\mathsf{M}, \mathsf{t}]$ may be introduced, but M must be the same action model as in the premise. Thus, these cases are just special cases of the third bullet in Definition 59. For the last two cases it follows from Lemma 58 that also these two preserve d-subformulas. □

Definition 62. *Given a tableau branch Θ and a prefix σ that occurs on Θ, let*

$$T^{\Theta}(\sigma) := \{\varphi \mid \sigma\varphi \text{ is on } \Theta\}.$$

Lemma 63. *Let Θ be a branch of a tableau and σ a prefix occurring on it. Then the set $T^{\Theta}(\sigma)$ is finite.*

Proof. By Lemma 61, all formulas on Θ are d-subformulas of the root formula. Thus, the lemma follows if we can show that for all formulas φ, the set of d-subformulas of φ is finite. This can be proved by induction on $n = d(\varphi)$ given a fixed number of propositional variables N. For $n = 1$: It is obvious that there can only be N many different d-subformulas of φ, for all formulas φ with complexity $d(\varphi) = 1$. For the induction step, assume there are only finitely many d-subformulas of φ, for all φ with $d(\varphi) \leq n$. Given a formula φ with $d(\varphi) = n+1$, it is easy to see that any d-subformula of φ is also a d-subformula of a formula with d-complexity less than or equal to n or constructed from one of these. By induction there can only be finitely many of the first kind. For the second kind, we divide into cases depending on the structure of φ. It is easy to see that given finitely many formulas only finitely many new formulas can be constructed using the logical connectives and the K_a operators (since there are only finitely many a's). In the case of the action modalities, note that point 3 of Definition 59 only allows for finitely many domains of action models, and the limitation on the d-complexity of the preconditions ensures that we can only construct finitely many different action modalities. This completes the proof of the lemma. □

Definition 64. *Let Θ be a tableau branch and σ a prefix occurring on Θ, then define $m_\Theta(\sigma)$ by*

$$m_\Theta(\sigma) = \max\{d(\varphi) \mid \sigma\varphi \in \Theta\}.$$

Note that Lemma 63 justifies that this is well-defined. We can now adopt the method of [37] to show that **AM** tableaux always terminates.

Definition 65. *When a prefix τ has been introduced on a branch Θ by the rule $(\neg K_a)$ to a formula $\sigma\varphi$, we say that τ is* generated *by σ and denote it by $\sigma \prec_\Theta \tau$.*

Following this we can easily prove, as in [37], that:

Lemma 66. *If Θ is a tableau branch, then Θ is infinite if and only if there exist an infinite chain of prefixes on Θ*

$$\sigma_1 \prec_\Theta \sigma_2 \prec_\Theta \sigma_3 \prec_\Theta \dots$$

Proof. See [37]. □

Lemma 67. *Let Θ be a tableau branch and σ and τ two prefixes occurring on Θ. Then $\sigma \prec_\Theta \tau$ implies that $m_\Theta(\sigma) > m_\Theta(\tau)$.*

Proof. The proof carries through just as in [37], once it has been noted that the rules $([AM])$ and $(\neg[AM])$ decrease the d-complexity from the premise to the conclusion, and that none of these rules introduce new prefixes. □

As in [37] termination now easily follows from the lemmas 66 and 67:

Theorem 68 (Termination of the tableau system)**.** *Any tableau constructed for a $\mathcal{L}_{K\otimes}$-formula is finite.*

5.3.2 Soundness and completeness of the tableau system

Soundness is not hard to prove. The rules for the underlying multi modal logic are standard and easily seen to be sound. By the validity of the reduction axioms (5.9)–(5.13), the soundness of the rules ($[AM]$) and ($\neg[AM]$) follows.

The completeness for the underlying multi modal logic using only the rules involving this part of the language is already well known (see for instance [37]). Given an open saturated branch Θ, a canonical model $\mathcal{M}$ is constructed from the prefixes occurring on Θ and the accessibility relations are defined by which accessibility formulas $\sigma R_a \tau$ occur on Θ. The valuation of a propositional variable p is defined relative to which of σp and $\sigma\neg p$ (if any) occurs on Θ. It is then straightforward to prove a truth lemma stating that; for all prefixed formulas $\sigma\varphi$ on Θ,

$$\mathcal{M}, \sigma \models \varphi.$$

For our tableau system this construction and the formulation of the truth lemma are identical. However, instead of proving the truth lemma by induction on formula complexity, we prove it by induction on the d-complexity and add two new cases for $[\mathsf{M}, \mathsf{s}]\varphi$ and $\neg[\mathsf{M}, \mathsf{s}]\varphi$. These cases are, however, quite straightforward: Assume that $\sigma[\mathsf{M}, \mathsf{s}]\varphi$ occurs on Θ. Then by saturation of Θ, $\sigma t([\mathsf{M}, \mathsf{s}]\varphi)$ also occurs on Θ and since $t([\mathsf{M}, \mathsf{s}]\varphi)$ has less d-complexity than $\sigma[\mathsf{M}, \mathsf{s}]\varphi$, it follows by induction that $\mathcal{M}, \sigma \models t([\mathsf{M}, \mathsf{s}]\varphi)$. But then, by the validity of the reduction axioms (5.9)–(5.13), it follows that also $\mathcal{M}, \sigma \models [\mathsf{M}, \mathsf{s}]\varphi$. The case for $\neg[\mathsf{M}, \mathsf{s}]\varphi$ is similar. Thus we get:

Theorem 69. *The tableau system of Figure 5.1 is sound and complete with respect to the logic* ***AM****.*

5.4 A tableau for hybrid public announcement logic

In this section, we introduce a tableau system for **HPA**. It is both simpler and more complicated than the tableau system of the previous section. The simplification consist in looking purely at public announcement, whereas the complication consist in extending the underlying epistemic logic to a hybrid logic. The simplification shortens the proof of the lemmas 61 and 63 considerably, but the hybrid machinery makes us in need of a more advanced termination proof as in [37]. Our tableau system will be based on a small modification of the one in [37] (also shown i Figure 1.7 of Section 1.2.2), further extended with reduction axiom rules for public announcement. We reuse all of the terminology of Section 5.3. The tableau rules are given in Figure 5.2.

$$\frac{\sigma\neg\neg\varphi}{\sigma\varphi}\,(\neg\neg) \qquad \frac{\sigma\varphi\wedge\psi}{\begin{array}{c}\sigma\varphi\\ \sigma\psi\end{array}}\,(\wedge) \qquad \frac{\sigma\neg(\varphi\wedge\psi)}{\sigma\neg\varphi \;\big|\; \sigma\neg\psi}\,(\neg\wedge)$$

$$\frac{\sigma\neg K_a\varphi}{\begin{array}{c}\sigma R_a\tau\\ \tau\neg\varphi\end{array}}\,(\neg K_a)^1 \qquad \frac{\sigma K_a\varphi \quad \sigma R_a\tau}{\tau\varphi}\,(K_a) \qquad \frac{\sigma E\varphi}{\tau\varphi}\,(E)^1 \qquad \frac{\sigma\neg E\varphi}{\tau\neg\varphi}\,(\neg E)^2$$

$$\frac{\sigma @_i\varphi}{\begin{array}{c}\tau i\\ \tau\varphi\end{array}}\,(@)^1 \qquad \frac{\sigma\neg @_i\varphi}{\sigma\neg Ei \;\Big|\; \begin{array}{c}\tau i\\ \tau\neg\varphi\end{array}}\,(\neg @)^1 \qquad \frac{\sigma\varphi \quad \sigma i \quad \tau i}{\tau\varphi}\,(Id)$$

$$\frac{\sigma[\varphi]\psi}{\sigma t([\varphi]\psi)}\,([]) \qquad \frac{\sigma\neg[\varphi]\psi}{\sigma\neg t([\varphi]\psi)}\,(\neg[])$$

[1] The prefix τ is new to the branch. [2] The prefix τ is already on the branch.

Figure 5.2: Tableau rules for **HPA**.

In the rules ([]) and (¬[]), the operation t is defined via the reduction axioms for **HPA**, in the same way as in the previous section. Compared to [37] one rule has also been left out[10], and the rule (¬@) has been altered.[11] Both changes have been made to deal with the fact that nominals in our logic only partially denote states. The rules $(\neg K_a)$, (@), (¬@) and (E) are called *prefix generating* rules. The construction of a tableau is done with the constraints that no prefix generating rule is applied twice to the same premise on the same branch, and a formula is never added to the branch if it already occurs on that branch. Furthermore, to make the tableaux terminate, we introduce (as in [37]) a loop-checking mechanism. Before this we need the notion of an "urfarther".

Definition 70. *Given a branch Θ, the prefix τ is an* urfather[12] *of the prefix σ if τ is the earliest introduced prefix on Θ such that $T^\Theta(\sigma) \subseteq T^\Theta(\tau)$. We denote this by $u_\Theta(\sigma) = \tau$.*

[10] The rule (¬) of Figure 1.7 of Section 1.2.2 has been left out since it is no longer sound when nominals only partially denotes states.

[11] If $\neg @_i\varphi$ is true it either means that i does not denote a state (in which case $\neg Ei$ is true) or that i denotes a state and φ is true there. This justifies the modified (¬@) rule.

[12] This notion of an urfarther used here is called an inclusion urfarther in [37].

The construction of a **HPA** tableau is subject to the following constraint:

> *A prefix generating rule is only allowed to be applied to a formula $\sigma\varphi$ on a branch if σ is an urfather on that branch.*

5.4.1 Termination of HPA tableaux

As in the general action model case, we need an extended notion of subformulas based on the complexity measure c of Definition 56.

Definition 71. *A formula ψ is said to be a c-*subformula *of a formula φ if*

- $c(\psi) \leq c(\varphi)$
- *Every propositional variable and all the nominals that occur in ψ also occur in φ.*

In the case of **HPA** tableaux, the following can straightforwardly be proven:

Lemma 72. *For every tableau rule the c-complexity of the conclusion is less than the c-complexity of the premise.*

Lemma 73 (c-subformula property)**.** *Let $\mathcal{T}$ be a tableau with root formula $\sigma\varphi$. If the prefixed formula $\tau\psi$ occurs on $\mathcal{T}$, then ψ is a c-subformula of φ.*

Proof. By Lemma 72 and the fact that no rules can introduce new nominals or propositional variables, it is easy to check for all the rules that if they have c-subformulas as premises, the conclusions will also be c-subformulas.

Note that the rule $(\neg @)$ can only be applied if a prefixed formula $\tau\neg @_i\chi$ occurs on the tableau, in which case, by induction $c(\neg @_i\chi) \leq c(\varphi)$. Thus, it follows that $c(\neg Ei) \leq c(\varphi)$, and hence all formulas of the form $\tau\neg Ei$ occurring on $\mathcal{T}$ will, also be c-subformulas of the root formula φ. □

The following lemma is easier to prove in the case of **HPA**.

Lemma 74. *For all tableau branches Θ and prefixes σ occurring on Θ, the set $T^{\Theta}(\sigma)$ is finite.*

Proof. From Lemma 73 it follows that $T^{\Theta}(\sigma)$ is a subset of the set of all c-subformulas of the root formula of Θ. Thus the lemma follows if we can show that for all formulas φ, the set of c-subformulas of φ is finite. The proof of this is similar to the proof of Lemma 63, but easier, since the public announcement operator is not as complicated as the action modalities. □

We now extend the ordering $\prec_\Theta$ introduced in the previous section. Let Θ be a tableau branch. If a prefix τ has been introduced to the branch using a prefix generating rule on a formula of the form $\sigma\varphi$, we say that τ *is generated by* σ and write $\sigma \prec_\Theta \tau$. It is straightforward to show that Lemma 66 remains true in this case.

The rest of the proof of termination is identical to the proof of Theorem 5.4 in [37]. The only difference is that their notion of quasi-subformula has to be replaced by our notion of c-subformula. Thus we get that:

Theorem 75. *Any tableau constructed using the given rules for* ***HPA*** *is finite, and thus the logic* ***HPA*** *is decidable.*

5.4.2 Soundness and completeness of the tableau system for HPA

Again, as for **AM**, the proof of soundness is simple. The completeness is also almost as in [37]. The only modification needed is because nominals only partly denote in **HPA**. The reduction axiom rules are dealt with as for the tableau system for **AM**. Given an open saturated branch Θ, a model $\mathcal{M}^\Theta = \langle W^\Theta, R^\Theta, V^\Theta \rangle$ is constructed as in [37] by:

$$\begin{aligned} W^\Theta &= \{\sigma \mid \sigma \textit{ is an urfather on } \Theta\} \\ R_a^\Theta &= \{(\sigma, u_\Theta(\tau)) \in W^\Theta \times W^\Theta \mid \sigma R_a \tau \textit{ occurs on } \Theta\} \\ V^\Theta(x) &= \{u_\Theta(\sigma) \in W^\Theta \mid \sigma x \textit{ occurs on } \Theta\}, \textit{ for all } x \in \mathsf{PROP} \cup \mathsf{NOM}. \end{aligned}$$

For V to be well-defined, we have to make sure that $|V^\Theta(i)| \leq 1$ for all nominals i. If there were two different urfathers σ and τ and a nominal i, such that both σi and τi occurred on Θ, then using the saturation of the branch and the rule (Id), we would get that $T^\Theta(\sigma) = T^\Theta(\tau)$. However, since they were both urfathers, this would imply that $\sigma = \tau$, which is a contradiction. Thus V is well-defined. Now completeness follows from the following theorem:

Theorem 76. *Let Θ be an open saturated branch for the tableau system. For any formula $\sigma\varphi$ occurring on Θ, with σ an urfather, it holds that $\mathcal{M}^\Theta, \sigma \models \varphi$.*

Proof. The proof goes by induction on the complexity of φ. The basic cases follow from the definition of V^Θ. The cases $\varphi = K_a\psi$, $\varphi = \neg K_a\psi$, $\varphi = E\psi$, $\varphi = \neg E\psi$, and $\varphi = @_i\psi$ are as in [37].

In the case of $\varphi = \neg @_i\psi$ a little more work is required. Assume that $\sigma\neg @_i\varphi$ occurs on Θ and that σ is an urfather. Then by the closure of the rule

($\neg@$) either $\sigma\neg Ei$ or τi and $\tau\neg\varphi$ occur on the branch. In the first case, there can be no prefix σ' such that $\sigma' i$ is on Θ. This is because the rule $\neg E$ gives that $\sigma'\neg i$ is also on Θ, which contradicts the assumption that Θ is an open branch. But then there can be no state in $\mathcal{M}^\Theta$, which i denotes. Thus by the semantic $\mathcal{M}^\Theta, \sigma \models \neg @_i\varphi$. On the other hand if τi and $\tau\neg\varphi$ are on Θ for a prefix τ, then by urfather closure, also $u_\Theta(\tau)i$ and $u_\Theta(\tau)\neg\varphi$ are on Θ, which by the induction hypothesis gives that $\mathcal{M}^\Theta, u_\Theta(\tau) \models i$ and $\mathcal{M}^\Theta, u_\Theta(\tau) \models \neg\varphi$. Thus we get that $\mathcal{M}^\Theta, \sigma \models \neg @_i\varphi$. □

As a consequence of this we get that:

Theorem 77. *The tableau system of Figure 5.2 is sound and complete with respect to the logic* ***HPA****.*

5.5 Concluding remarks and further research

In this paper, we have presented two tableau systems; one for dynamic epistemic logic with action models and one for a hybrid public announcement logic (both without common knowledge). These were based on already existing tableau systems to which we simply added tableau rules corresponding to the reduction axioms of the two logics. Following this, we showed that the method used to prove termination in [37], can also be extended to our new tableau systems.

There are already tableau systems for **PA**, [49] and [14], of which the one in [14] is shown to be optimal with respect to complexity. However, these only work for **PA** and cannot be generalized to other DELs in an obvious way. The aim of this paper has not been to construct complexity optimal tableau systems, but to show how tableau systems can be obtained in a more general way for various DELs.

Due to the unknown complexity status of **AM** and the problem of how exactly to measure the length of formulas, it is unknown whether the tableau method here presented is optimal with respect to complexity. However, it does seem to provide some kind of exponential upper bound. In the case of **HPA**, the underlying hybrid logic has an EXPTIME complexity as it contains the global modality [144]. Again, the system here presented seems also to yield an exponential upper bound in this case. The exact complexity details are left for further research.

Presently, there exist no tableau systems (known to the author) for DELs extended with common knowledge, and, due to the lack of reduction axioms, our method cannot be used. However, in [157] it is shown that a generalization of common knowledge called "relativized common knowledge" allows for reduction axioms for the public announcement operator. Thus, if tableau systems can be constructed for a multi-modal logic extended with relativized common knowledge, the method here presented may be extendable to give a terminating tableau system for a public announcement logic with a form of common knowledge. The relativized common knowledge resembles the until operator from temporal logics interpreted over arbitrary Kripke frames. Hence, it might be possible to extend tableau systems from temporal logics to public announcement logics with relativized common knowledge.

An even more general setting for reduction axioms has been given by Barteld Kooi in [105]. A further direction of research would be to extend the methods presented here in order to make them work in that setting.

A final matter of concern is the choice to only deal with logics where the underlying modal logic is multi modal **K**. In epistemic logics, you usually add extra requirements to the agents' accessibility relations, which causes the underlying modal logic to change into for instance **S5** or **KD45**. It is therefore important to be able to extend the presented tableau systems to also deal with these cases. The methods here presented are based on the paper [37], which fortunately has a follow-up paper ([36]) that deals with the problems of adding extra conditions to the accessibility relations. It seems possible to use that work in connection with the tableau systems presented in this paper, but the exact details are left for further research.

Acknowledgment

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Chapter 6

A Logic-Based Approach to Pluralistic Ignorance

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Abstract: “Pluralistic ignorance” is a phenomenon mainly studied in social psychology. Viewed as an epistemic phenomenon, one way to define it is as a situation where “*no one believes, but everyone believes that everyone else believes*”. In this paper various versions of pluralistic ignorance are formalized using epistemic/doxastic logic (based on plausibility models). The motive is twofold. Firstly, the formalizations are used to show that the various versions of pluralistic ignorance are all consistent, thus there is nothing in the phenomenon that necessarily goes against logic. Secondly, pluralistic ignorance, is on many occasions, assumed to be fragile. In this paper, however, it is shown that pluralistic ignorance need not be fragile to announcements of the agents' beliefs. Hence, to dissolve pluralistic ignorance in general, something more than announcements of the subjective views of the agents is needed. Finally, suggestions for further research are outlined.

6.1 Introduction

Pluralistic ignorance is a term from the social and behavioral sciences going back to the work of Floyd H. Allport and Daniel Katz [104].[1] [107] (pp.

[1] See [122] for more on the coining of the term “pluralistic ignorance”.

388-89) define pluralistic ignorance as a situation where "*no one believes, but everyone believes that everyone else believes*". Elaborated, pluralistic ignorance is the phenomenon where a group of people shares a false belief about the beliefs, norms, actions, or thoughts of the other group members. It is a social phenomenon where people make systematic errors in judging other people's private attitudes. This makes it an important notion in understanding social life. However, pluralistic ignorance is a term used to describe many different phenomena that all share some common features. Therefore, there are many different definitions and examples of pluralistic ignorance and a few of the most common of these will be presented in Section 6.2.

Pluralistic ignorance has been approached by formal methods before [46, 91], but to the knowledge of the author, [91] is the only paper that takes a logic-based approach. [91] models pluralistic ignorance using formal learning theory and logic. In this paper, the tool will be classical modal logic in the form of doxastic/epistemic logic. In Section 6.3 we introduce a doxastic/epistemic logic based on the plausibility models presented in [17]. The reason for choosing this framework instead of, for instance, the multi-modal logic **KD45**, is that **KD45** cannot straightforwardly be combined with public announcements.[2] Since one of the aspects of pluralistic ignorance studied in this paper is the question of what it takes to dissolve the phenomenon, we need to be able to talk about the dynamics of knowledge and beliefs. Public announcements are the simplest form of actions that can affect the beliefs and knowledge of the agents and they therefore serve the purpose of this paper perfectly.

After having presented the formal framework in Section 6.3, it is possible in Section 6.4 to give a formal analysis of the different versions of pluralistic ignorance. We will give several different formalizations of pluralistic ignorance and discuss whether they are satisfiable or not. Afterwards, we will look at what it takes to dissolve pluralistic ignorance and show that, in general, something more than mere announcements of agents' true beliefs is needed. Since the logical approach to pluralistic ignorance is still very limited, there is ample opportunity for further research and several suggestions will be discussed in Section 6.5. The paper ends with a concise conclusion.

[2] Public announcement of a formula φ corresponds, in the model theory of modal logic, to the operation of going to the submodel only containing worlds where φ was true. However, the class of frames underlying the logic **KD45** is not closed under taking submodels, since seriality is not preserved when going to submodels. When combined with public announcement the logic **KD45** actually turns into the logic **S5**.

6.2 Examples of pluralistic ignorance

Examples of pluralistic ignorance are plentiful in the social and behavioral sciences literature. One example is the drinking of alcohol on (American) college campuses. Several studies have shown that many students feel much less comfortable with drinking than they believe the average college student does [130]. In other words, the students do not believe that drinking is at all enjoyable, but they still believe that all of their fellow students believe drinking to be quite enjoyable. Another classical example is the classroom example in which, after having presented the students with difficult material, the teacher asks them whether they have any questions. Even though most students do not understand the material they may not ask any questions. All the students interpret the lack of questions from the other students as a sign that they understood the material, and to avoid being publicly displayed as the stupid one, they dare not ask questions themselves. In this case the students are ignorant with respect to some facts, but believe that the rest of the students are not ignorant about the facts.

A classical made-up example is from Hans Christian Andersen's fable "The Emperor's New Clothes" from 1837. Here, two impostors sell imaginary clothes to an emperor claiming that those who cannot see the clothes are either not fit for their office or just truly stupid. Not wanting to appear unfit for his office or truly stupid, the Emperor (as well as everyone else) pretends to be able to see the garment. No one *personally* believes the Emperor to have any clothes on. They do, however, believe that everyone else believes the Emperor to be clothed. Or alternatively, everyone is ignorant to whether the Emperor has clothes on or not, but believes that everyone else is not ignorant. Finally, a little boy cries out: "but he has nothing on at all!" and the pluralistic ignorance is dissolved.

What might be clear from these examples is that pluralistic ignorance comes in many versions. A logical analysis of pluralistic ignorance may help categorize and distinguish several of these different versions. Note that these examples were all formulated in terms of beliefs, but pluralistic ignorance is often defined in the term of norms as well. [46] define pluralistic ignorance as "a situation where a majority of group members privately reject a norm, but assume (incorrectly) that most others accept it".

Misperceiving other people's norms or beliefs can occur without it being a case of pluralistic ignorance. Pluralistic ignorance is the case of systematic errors in norm/belief estimation of others. Thus, pluralistic ignorance is a genuine social phenomenon and not just people holding wrong beliefs about other people's norms or beliefs [122]. This might be the reason why pluralistic ignorance is often portrayed as a fragile phenomenon. Just one public announcement of a private belief or norm will resolve the case of pluralistic ignorance. In "The Emperor's New Clothes" a little boy's outcry is enough to dissolve the pluralistic ignorance. If, in the classroom example, one student dares to ask a question (and thus announces his academic ignorance) the other students will surely follow with questions of their own. In some versions of pluralistic ignorance, the mere awareness of the possibility of pluralistic ignorance is enough to suspend it. This fragility might not always be the case and, as we shall see, there is nothing in the standard definitions of pluralistic ignorance that forces it to be a fragile phenomenon.

6.3 Plausibility models: A logical model of belief, knowledge, doubt, and ignorance

We will model knowledge and beliefs using modal logic. More specifically, we will be using the framework of [17]. This section is a review of that framework. We will work in a multi-agent setting and thus, assume a finite set of agents $\mathbb{A}$ to be given. Furthermore, we also assume a set of propositional variables PROP to be given. The models of the logic will be special kinds of Kripke models called plausibility "models":

Definition 78. *A plausibility model is a tuple $\mathcal{M} = \langle W, (\leq_a)_{a\in\mathbb{A}}, V\rangle$, where W is a non-empty set of possible worlds/states, $\leq_a$ is a locally connected converse well-founded preorder on W for each $a \in \mathbb{A}$, and V is a valuation that to each $p \in \mathsf{PROP}$ assigns a subset of W.*

A relation is a *locally connected converse well-founded preorder* on W if it is locally connected (wherever x and y are related and y and z are related, then x and z are also related), converse well-founded (every non-empty subset of W has a maximal element), and is a preorder (it is reflexive and transitive). In the following we will sometimes refer to the plausibility models simply as models.

The intuition behind plausibility models is that the possible worlds represent different ways the world might be. That $w \leq_a v$ for an agent a means that agent a thinks that the world v is at least as possible as world w, but a cannot distinguish which of the two is the case. The relation $\leq_a$ will be used to define what agent a believe. To define what agent a knows we introduce an equivalence relation $\sim_a$ defined by:

$$w \sim_a v \quad \text{if, and only if} \quad w \leq_a v \text{ or } v \leq_a w$$

The intuition behind $w \sim_a v$ is that for all that agent a knows, she cannot distinguish between which of the worlds w and v is the case. Given an agent a and a world w, the set $|w|_a = \{v \in W \mid v \sim_a w\}$ is the information cell at w of agent a and represent all the worlds that agent a considers possible at the world w. In other words, this set encodes the hard information of agent a at the world w.

Based on the introduced notions, we can now define knowledge and beliefs. Let K_a and B_a be modal operators for all agents $a \in \mathbb{A}$. We read $K_a\varphi$ as "agent a knows that φ" and $B_a\varphi$ as "agent a believes that φ". We specify the formal language $\mathcal{L}$, which we will be working with, by the following syntax:

$$\varphi \ ::= \ p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K_a\varphi \mid B_a\varphi,$$

where $p \in \mathsf{PROP}$ and $a \in \mathbb{A}$. The logical symbols $\top, \bot, \vee, \rightarrow, \leftrightarrow$ are defined in the usual way. The semantics of the logic is then defined by:

Definition 79. *Given a plausibility model $\mathcal{M} = \langle W, (\leq_a)_{a\in\mathbb{A}}, V\rangle$ and a world $w \in W$ we define the semantics inductively by:*

$$\begin{array}{lll}
\mathcal{M}, w \models p & \textit{iff} & w \in V(p) \\
\mathcal{M}, w \models \neg\varphi & \textit{iff} & \textit{it is not the case that } \mathcal{M}, w \models \varphi \\
\mathcal{M}, w \models \varphi \wedge \psi & \textit{iff} & \mathcal{M}, w \models \varphi \textit{ and } \mathcal{M}, w \models \psi \\
\mathcal{M}, w \models K_a\varphi & \textit{iff} & \textit{for all } v \in |w|_a,\ \mathcal{M}, v \models \varphi \\
\mathcal{M}, w \models B_a\varphi & \textit{iff} & \textit{for all } v \in max_{\leq_a}(|w|_a),\ \mathcal{M}, v \models \varphi,
\end{array}$$

where $max_{\leq_a}(S)$ is the set of maximal elements of S with respect to the relation $\leq_a$. We say that a formula φ is satisfiable if there is a model $\mathcal{M}$ and a world w in $\mathcal{M}$ such that $\mathcal{M}, w \models \varphi$. A formula φ is valid if for all models $\mathcal{M}$ and all worlds w in $\mathcal{M}$, $\mathcal{M}, w \models \varphi$.

Note, that the semantics make $K_a\varphi \rightarrow B_a\varphi$ valid. In the framework of [17] other notions of beliefs are also introduced. The first is *conditional beliefs*; $B_a^{\varphi}\psi$ expresses that agent a believes that ψ was the case, if she learned that φ was the case. The semantics of this modality is:

$$\mathcal{M}, w \models B_a^{\varphi}\psi \quad \text{iff} \quad \text{for all } v \in max_{\leq_a}(|w|_a \cap [\![\varphi]\!]_{\mathcal{M}}),\ \mathcal{M}, v \models \psi,$$

where $[\![\varphi]\!]_{\mathcal{M}}$ is the set of worlds in $\mathcal{M}$ where φ is true. Another notion of belief is *safe belief* for which we use $\Box_a$. The semantics of this modality is:

$$\mathcal{M}, w \models \Box_a\varphi \quad \text{iff} \quad \text{for all } v \in W,\ \text{if } w \leq_a v,\ \text{then } \mathcal{M}, v \models \varphi.$$

Note, this is the usual modality defined from the relation $\leq_a$. Since $\leq_a$ is reflexive, $\Box_a\varphi \rightarrow \varphi$ is valid. Hence, safe belief is a very strong notion of belief (or a weak notion of knowledge). Because a central aspect of pluralistic ignorance is people holding *wrong* beliefs, safe belief is not a suitable notion. Yet another notion of belief, that also implies truth, is *weakly safe belief* $\Box_a^{\text{weak}}$ given by the following semantics:

$$\mathcal{M}, w \models \Box_a^{\text{weak}}\varphi \quad \text{iff} \quad \mathcal{M}, w \models \varphi \text{ and for all } v \in W,\ \text{if } w <_a v,\ \text{then } \mathcal{M}, v \models \varphi,$$

where $<_a$ is defined by; $w <_a v$ if and only if $w \leq_a v$ and $v \not\leq_a w$. Finally, [17] define *strong belief* Sb_a by

$$Sb_a\varphi \text{ iff } B_a\varphi \wedge K_a(\varphi \rightarrow \Box_a\varphi).$$

In addition to the several notions of belief, [17] also discuss several ways of updating knowledge and beliefs when new information comes about. These are *update*, *radical upgrade*, and *conservative upgrade* and can be distinguished by the trust that is put in the source of the new information. If the source is known to be infallible, it should be an update. If the source is highly reliable, it should be a radical upgrade and if the source is just barely trusted, it should be a conservative upgrade. In this paper we are interested in what it takes to dissolve pluralistic ignorance and since update is the "strongest" way of updating knowledge and beliefs, we will focus on this. We will also refer to this way of updating as *public announcement*.

We introduce operators $[!\varphi]$, and add to the syntax the clause that for all formulas φ and ψ, $[!\varphi]\psi$ is also a formula. $[!\varphi]\psi$ is read as "after an announcement of φ, ψ is true". Semantically, a public announcement of φ will result in a new plausibility model where all the $\neg\varphi$-worlds have been removed, and the truth of ψ is then checked in this new model. These intuitions are made formal in the following definition:

Definition 80. *Given a plausibility model $\mathcal{M} = \langle W, (\leq_a)_{a\in\mathbb{A}}, V\rangle$ and a formula φ, we define a new model $\mathcal{M}_{!\varphi} = \langle W', (\leq'_a)_{a\in\mathbb{A}}, V'\rangle$ by,*

$$\begin{array}{rcl} W' & = & \{w \in W \mid \mathcal{M}, w \models \varphi\} \\ \leq'_a & = & \leq_a \cap (W' \times W') \\ V'(p) & = & V(p) \cap W' \end{array}$$

The semantics of the public announcement formulas are then given by:

$$\mathcal{M}, w \models [!\varphi]\psi \quad \textit{iff} \quad \textit{if } \mathcal{M}, w \models \varphi \textit{ then } \mathcal{M}_{!\varphi}, w \models \psi.$$

Finally, we add to the framework of [17] the two notions of ignorance and doubt. These are notions rarely discussed in the literature on epistemic/doxastic logic. However, ignorance is discussed in [159]. On the syntactic level we add two new operators I_a and D_a for each agent $a \in \mathbb{A}$. The formula $I_a\varphi$ is read as "agent a is ignorant about φ" and $D_a\varphi$ is read as "agent a doubts whether φ". The semantics of these operators are defined from the semantics of the knowledge operator and the belief operator:

Definition 81. *The operators I_a and D_a are defined by the following equivalences:*

$$\begin{array}{rcl} I_a\varphi & := & \neg K_a\varphi \wedge \neg K_a\neg\varphi \\ D_a\varphi & := & \neg B_a\varphi \wedge \neg B_a\neg\varphi. \end{array}$$

Note that, since $K_a\varphi \to B_a\varphi$ is valid, $D_a\varphi \to I_a\varphi$ is also valid.

6.4 Modeling pluralistic ignorance

Based on the logic introduced in the previous section, we will now formalize different versions of pluralistic ignorance that are all consistent. Then, we will discuss whether these formalizations make pluralistic ignorance into a fragile phenomenon.

6.4.1 Formalizations and consistency of pluralistic ignorance

As discussed in Section 6.2, there are many ways of defining pluralistic ignorance and in this section we attempt to formalize a few of these. We will also discuss whether these formalizations lead to consistent concepts in the sense that the formalizations are satisfiable formulas.

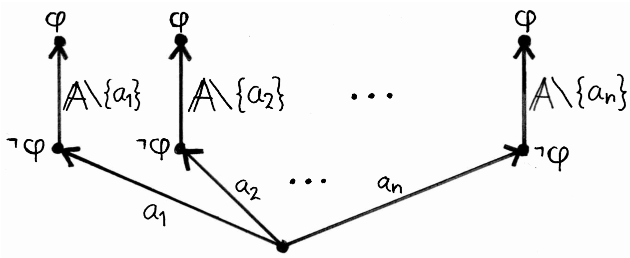

Figure 6.1: A plausibility model where (6.1) is satisfiable at the root.

Firstly, we assume that pluralistic ignorance is a situation where no agent believes φ, but every agent believes that everyone else believes φ. This can easily be formalized as:

$$\bigwedge_{a\in\mathbb{A}}\Big(\neg B_a\varphi \wedge \bigwedge_{b\in\mathbf{A}\setminus\{a\}} B_aB_b\varphi\Big) \tag{6.1}$$

For boolean formulas φ[3], (6.1) is satisfiable since a plausibility model can easily be constructed such that it contains a possible world that satisfies it. Such a model is given in Figure 6.1, where we assume that the set of agents is $\mathbb{A} = \{a_1, a_2, ..., a_n\}$. In the following, when drawing models like this one, an arrow from a state w to a state v labeled by a_i will represent that $w <_{a_i} v$ holds in the model. An arrow from w to v labeled by a set $B \subseteq \mathbb{A}$ represent that $w <_b v$ for all $b \in B$. The full plausibility relations of the model will be the reflexive transitive closures of the relations drawn in the pictures. When a formula φ appears next to a state it means that φ is true at that state.

There are also formulas φ for which (6.1) is unsatisfiable, take for instance φ to be $B_b\psi$ or $\neg B_b\psi$ for any agent $b \in \mathbb{A}$ and any formula ψ. It cannot be the case that agent a does not believe that agent b believes that ψ, but at the same time a believes that b believes that b believes that ψ, i.e. (6.1) is unsatisfiable when φ is $B_b\psi$ or $\neg B_b\psi$ because $\neg B_aB_b\psi \wedge B_aB_bB_b\psi$ and $\neg B_a\neg B_b\psi \wedge B_aB_b\neg B_b\psi$ are unsatisfiable. In the following, when discussing pluralistic ignorance as defined by (6.1) we will therefore assume that φ is a boolean formula.

If belief is replaced by strong belief, such that (6.1) becomes

$$\bigwedge_{a\in\mathbb{A}}\Big(\neg Sb_a\varphi \wedge \bigwedge_{b\in\mathbf{A}\setminus\{a\}} Sb_aSb_b\varphi\Big), \tag{6.2}$$

[3] A formula is boolean if it constructed solely from propositional variables and the logical connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$.

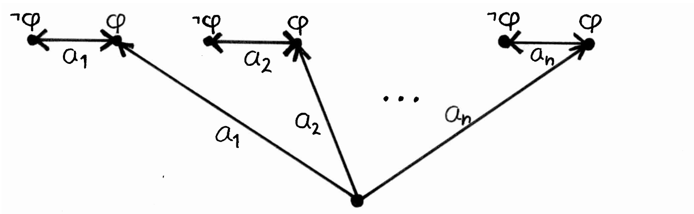

Figure 6.2: A plausibility model where (6.3) and (6.4) are satisfiable at the root.

pluralistic ignorance remains satisfiable for boolean formulas, which is testified by Figure 6.1 again. Furthermore, (6.2) is also not satisfiable if φ is of the form $Sb_b\psi$ or $\neg Sb_b\psi$ for a $b \in \mathbb{A}$. However, if we use safe belief and weak safe belief instead of belief in (6.1), pluralistic ignorance becomes unsatisfiable. This is obvious since both safe belief and weak safe belief implies truth.

In the classroom example of Section 6.2, a better definition of pluralistic ignorance may be obtained using the ignorance operator. This leads to the following definition of pluralistic ignorance:

$$\bigwedge_{a \in \mathbb{A}} \Big(I_a\varphi \wedge \bigwedge_{b \in \mathbf{A} \setminus \{a\}} B_a \neg I_b \varphi \Big) \tag{6.3}$$

This formula expresses a case where all the agents are ignorant about φ, but believe that all the other agents are not ignorant about φ. Instead of ignorance, doubt could be used as well, providing yet another definition of pluralistic ignorance:

$$\bigwedge_{a \in \mathbb{A}} \Big(D_a\varphi \wedge \bigwedge_{b \in \mathbf{A} \setminus \{a\}} B_a \neg D_b \varphi \Big) \tag{6.4}$$

Note that, since $D_a\varphi \rightarrow I_a\varphi$, (6.4) implies (6.3).

The two definitions of pluralistic ignorance (6.3) and (6.4) are also easily seen to be satisfiable for boolean formulas φ. This is made apparent by Figure 6.2. Now, however, formulas of the form $B_b\varphi$, for $b \in \mathbb{A}$, can also be subject to pluralistic ignorance. It is possible that agent a can doubt whether agent b believes φ and at the same time believe that agent b does not doubt whether he himself /agent b believes φ.

In (6.3) and (6.4) we can also replace the belief operator by the strong belief operator and obtain the following versions of pluralistic ignorance:

$$\bigwedge_{a \in \mathbb{A}} \Big(I_a\varphi \wedge \bigwedge_{b \in \mathbf{A} \setminus \{a\}} Sb_a \neg I_b \varphi \Big) \tag{6.5}$$

$$\bigwedge_{a \in \mathbb{A}} \Big(D_a\varphi \wedge \bigwedge_{b \in \mathbf{A} \setminus \{a\}} Sb_a \neg D_b \varphi \Big) \tag{6.6}$$

These new definitions of pluralistic ignorance are consistent as they are satisfied at the root of the model in Figure 6.2. We still cannot obtain versions of (6.3) and (6.4) with safe belief and weak safe belief for the same reason as before.

It seems obvious that we can formalize even further versions of pluralistic ignorance within this framework. Thus, using the logic introduced in Section 6.3, we can characterize and distinguish many different versions of pluralistic ignorance. Furthermore, all the definitions (6.1)-(6.6) were satisfiable, which seems to entail that the concept of pluralistic ignorance is not inconsistent.

6.4.2 The fragility of pluralistic ignorance

After having formalized different versions of pluralistic ignorance, we can ask whether any of the definitions entail that pluralistic ignorance is a fragile phenomenon. However, first of all we need to spell out what we mean by a fragile phenomenon. The question of whether pluralistic ignorance is fragile or not reduces to the question of what it takes to dissolve it. We will regard pluralistic ignorance as dissolved only when none of the agents have wrong beliefs about the other agents' beliefs anymore. The way agents can change their beliefs, will in this section be modeled by the $[!\varphi]$ operators of Section 6.3.

For the time being, we fix pluralistic ignorance to be defined as (6.1). According to several descriptions of pluralistic ignorance, it should be dissolved if just one agent announces his true beliefs. If the formula $!\neg B_b\varphi$ is announced, it naturally follows that $\bigwedge_{a\in\mathbb{A}} B_a \neg B_b \varphi$. However, this does not dissolve the pluralistic ignorance since all agents might keep their wrong beliefs about any other agent than b. In other words, a model satisfying (6.1) can be constructed such that after the announcement of $!\neg B_b\varphi$ it still holds that $\bigwedge_{a\in\mathbb{A}} \Big(\bigwedge_{c\in\mathbb{A}\setminus\{a,b\}} B_a B_c \varphi \Big)$.

It turns out that there is nothing in the definition (6.1) that prevents the wrong beliefs of the agents from being quite robust. Even if everybody except an agent c announces that they do not believe φ, all the agents might still believe that c believes φ. Using a formula of $\mathcal{L}$ we can define a notion of robustness in the following way: *agent a robustly believes that the group of agents $B \subseteq \mathbb{A}\setminus\{a\}$ believes φ* if

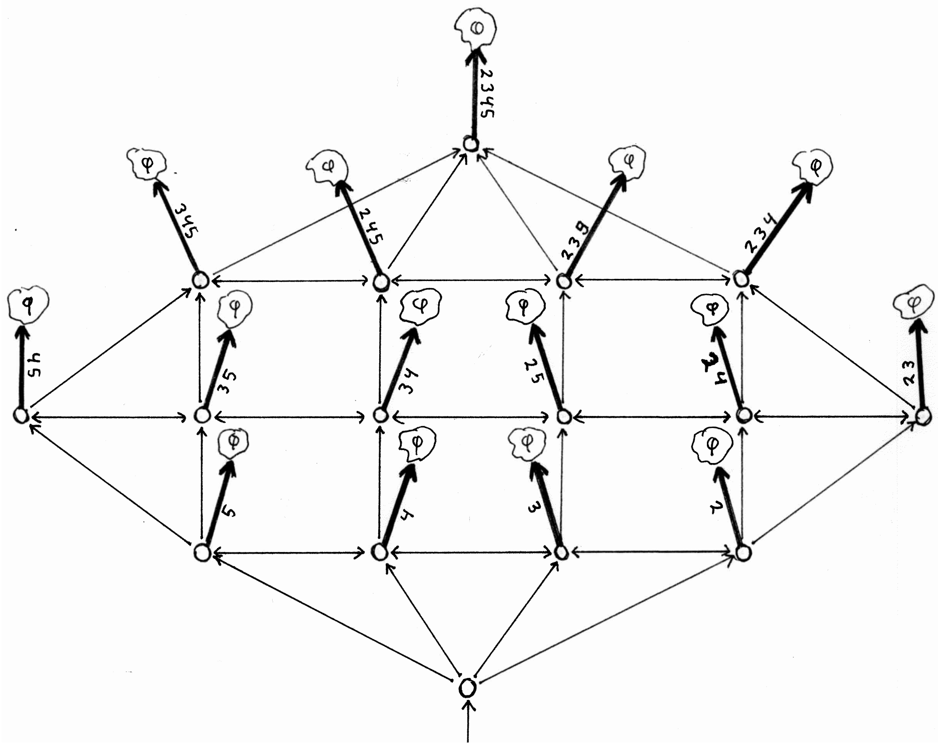

Figure 6.3: A robust model where agent 1 believes $\neg\varphi$ and has a strong robust belief that the agents 2, 3, 4, and 5 believe φ. The worlds marked with "$\circ$" are worlds where $\neg\varphi$ is true and the "clouds" marked with φ are collections of worlds where φ is true all over. The arrows not marked with numbers represnt the plausibility raltion for agent 1 only.

$$\bigwedge_{C\subseteq B}\Big([!\neg B_c\varphi]_{c\in C}\big(\bigwedge_{b\in B\setminus C} B_aB_b\varphi\big)\Big), \tag{6.7}$$

where $[!\neg B_c\varphi]_{c\in C}$ is an abbreviation for $[!\neg B_{c_1}\varphi][!\neg B_{c_2}\varphi]...[!\neg B_{c_k}\varphi]$, when $C=\{c_1,c_2,...,c_k\}$.[4] An example of a model where agent 1 believes $\neg\varphi$ and robustly believes that the agents $\{2,3,4,5\}$ believe φ is shown in Figure 6.3.

[4]An alternative to (6.7) is

$$\bigwedge_{C\subseteq B}\Big([!\bigwedge_{c\in C}\neg B_c\varphi]\big(\bigwedge_{b\in B\setminus C} B_aB_b\varphi\big)\Big),$$

however, the two are not equivalent. We will not go into a discussion of which definition is preferable.

If in the model of Figure 6.3 it is announced that $\neg B_2\varphi$ then all the $\circ$-worlds that have an outgoing arrow marked with 2 will be removed. Thus, the new "top" $\circ$-world will be the $\circ$-world with an outgoing arrow marked 345, which means that agent 1 still believes that the agents 3, 4, and 5 believe φ. Note that if it is announced that $\neg B_1\varphi$, what happens depends on the plausibility relation for agent 1 within the φ-clouds, and this plausibility relation is unspecified in the model of Figure 6.3. However, one could define the plausibility relation of agent 1 within the φ-clouds such that nothing happens to the model if $\neg B_1\varphi$ is announced.

Another way of looking at the formula (6.7) is that it describes a situation where agent a believes that all the other agents' beliefs about φ are independent; maybe they all believe φ for different reasons. Thus, learning about some agents' beliefs about φ tells a nothing about what the other agents believe about φ.

With robustness defined by (6.7), pluralistic ignorance is consistent with all the agents having wrong robust beliefs about the other agents' beliefs. Taking disjoint copies of the model in Figure 6.3 for each agent and joining the roots shows that:

Proposition 82. *Pluralistic ignorance in form of* (6.1) *is consistent with that all the agents $a \in \mathbb{A}$, robustly believes that the group of agents $\mathbb{A}\setminus\{a\}$ believe φ.*

Another way of interpreting this result is that announcements of the true beliefs of some of the involved agents are not enough to dissolve pluralistic ignorance. Either all the agents need to announce their true beliefs or new information has to come from an outside trusted source. Thus, announcements of the forms $!B_a\varphi$ or $!\neg B_a\varphi$ are not guaranteed to dissolve pluralistic ignorance. However, a public announcement of $!\neg\varphi$ in the model of Figure 6.3 will remove the pluralistic ignorance. But an announcement of the form $!\neg\varphi$ (or $!\varphi$) is precisely an announcement from a trusted outsider. An agent a in $\mathbb{A}$ can only announce formulas of the form $!B_a\psi$ or $!\neg B_a\psi$.

What turns pluralistic ignorance into a fragile phenomenon in most cases, is the fact that the agents consider the other agents' beliefs not to be independent as is the case if (6.7) is satisfied. In other words, pluralistic ignorance in the fragile form occurs mainly when the beliefs of the involved agents are correlated. This fits well with the view that pluralistic ignorance is a genuine social phenomenon as claimed by [122].

Proposition 82 only regards pluralistic ignorance as defined by (6.1). How-

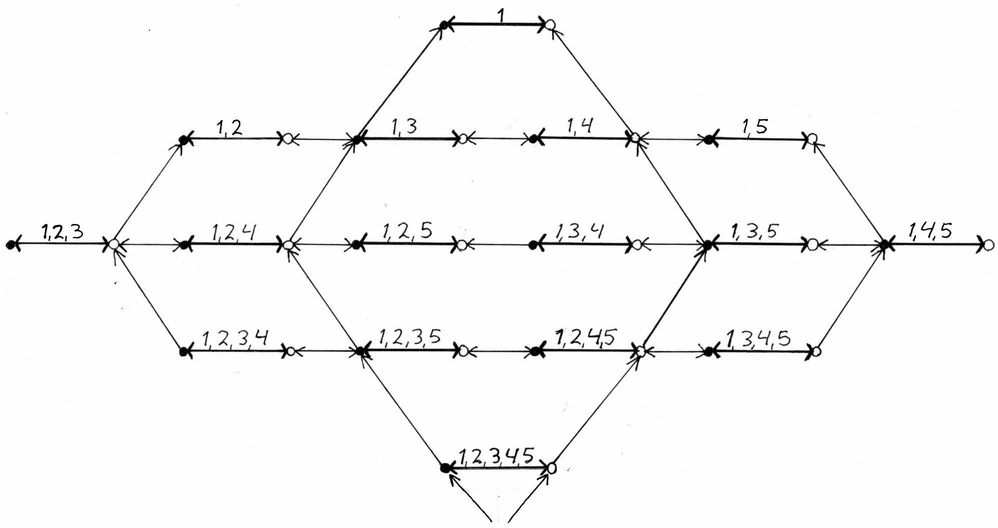

Figure 6.4: A robust model where agent 1 doubts whether φ and has a strong robust belief in that the agents 2, 3, 4, and 5 do not doubt whether φ. The worlds marked with "$\circ$" are worlds where $\neg\varphi$ is true and the worlds marked with "$\bullet$" are worlds where φ is true. The arrows with no numbers on are arrows for agent 1. Remeber that the full plausibility relations of the model are the reflexive transitive closures of the arrows in the pictures.

ever, for the definitions (6.3) and (6.4) similar results hold. Neither of the definitions (6.3) and (6.4) entail that pluralistic ignorance is fragile to public announcements of doubts ($[!D_b\varphi]$) or ignorance ($[!I_b\varphi]$). We can construct a new model, similar to the one in Figure 6.3, in which an agent a doubts whether φ but has a strong robust belief in that all the agents in $\mathbb{A}\setminus\{a\}$ do not doubt whether φ (and the same goes for ignorance). This new model is shown in Figure 6.4.

When it comes to the definitions of pluralistic ignorance based on strong beliefs (6.2), (6.5), and (6.6), something interesting happens. In the model of Figure 6.3 agent 1 does not have a strong belief that the other agents have strong beliefs in φ. For instance, there is a state where $B_1Sb_5\varphi$ and $Sb_5\varphi$ are true, but $\Box_1Sb_5\varphi$ is not true. The same issue occurs in the model of Figure 6.4. It is still unknown whether robust models can be constructed such that they satisfy the strong belief versions of pluralistic ignorance as defined by (6.2), (6.5), and (6.6). Thus, it is left for further research whether there are strong belief versions of pluralistic ignorance that are not fragile. There are also several other questions for further research, which we will turn to now.

6.5 Further research on logic and pluralistic ignorance

We have given several consistent formalizations of pluralistic ignorance, but there still seems to be more possible variations to explore. Furthermore, we have been working within one specific framework, and the question remains whether there are other natural frameworks in which all formalizations of pluralistic ignorance become inconsistent. This would be highly unexpected though. Another question regarding formalizations of pluralistic ignorance in different frameworks is whether it changes the fragility of the phenomenon. This is still an open question.

Even though pluralistic ignorance need not be fragile, neither as a "real life" phenomenon nor according to the formalizations given in this paper, it seems that the really interesting cases occur when pluralistic ignorance is, in fact, fragile. Whether pluralistic ignorance is fragile appears to be closely related to it being a genuine social phenomenon; the dependence between agents' beliefs is what makes pluralistic ignorance fragile. Thus, the real interesting question for further research is how agents' beliefs are interdependent in the case of pluralistic ignorance and how best to model this in logic. In answering this question, a shift in focus from what it takes to dissolve pluralistic ignorance, to what it takes for pluralistic ignorance to arise, seems natural.

6.5.1 Informational Cascades: How pluralistic ignorance comes about and how it vanishes

An agent's beliefs may depend on other agents' beliefs in many ways; one way is through testimony of facts by other agents in which the agent trusts. Modeling trust and testimony is for instance done in [99]. Another way in which agents' beliefs may depend on each other could be through a common information source [23]. Yet another way is through informational cascades.

Informational cascades are phenomena widely discussed in the social sciences [23, 112] and the therm was introduced by [22]. Assume that some agents are supposed to act one at a time in a given order and that their actions depend on a private information source as well as the information obtained by observing the actions of the agents already having acted. When actions are performed sequentially and agents start to ignore their private information and instead base their actions merely on information obtained from the actions of the previous agents, an *informational cascade* has occurred. If the actions of the first people in the cascade oppose to their private beliefs and the remaining people join in with the same actions (also oppose to their private beliefs) the result might be a case of pluralistic ignorance. However, informational cascades are also fragile [22] and opposite cascades may occur, thus eliminating pluralistic ignorance again.

These kinds of informational cascades, which have been shown to occur in numerous of places, may very well be the cause of pluralistic ignorance. Hence, logical frameworks that can model informational cascades might also be suited to model pluralistic ignorance. To the knowledge of the author, the only paper on logic-based models of informational cascades is [99] and it may very well be possible to model pluralistic ignorance in that framework. However, further work on the logics of informational cascades is still to come.

6.5.2 Private versus public beliefs – the need for new notions of group beliefs

The concept of pluralistic ignorance, regardless of which version one adopts, seems to hint at the need for new notions of common knowledge/beliefs. Pluralistic ignorance can be viewed as a social phenomenon where everybody holds a private belief in φ, but publicly display a belief in $\neg\varphi$ and thus contribute to a "common belief" ("public belief" might be a better world) in $\neg\varphi$. Due to the usual definition of common belief (everybody believes φ and everybody believes that everybody believes φ and ...), a common belief in $\neg\varphi$ leads to private belief in $\neg\varphi$ for all agents in the group, but this is exactly the thing that fails in social epistemic scenarios involving pluralistic ignorance. Hence, a new notion of common group belief seems to be needed. In general, there are various ways in which group beliefs can be related to the beliefs of individuals of the group. Thus, a logic that distinguishes between private and public beliefs or contains new notions of common beliefs may help model pluralistic ignorance more adequately. Once again, this is left for further research.

6.5.3 How agents act

The way agents act in cases of pluralistic ignorance also seems to play an important role. The reason why most students believe that other students are comfortable with drinking might be that they observe the other students drinking heavily. In the classroom example students are also obtaining their wrong beliefs based on the observation of others. Furthermore, focusing on actions might also tell us something about how pluralistic ignorance evolves in the first place.

Therefore, a logic combining beliefs and actions might be the natural tool for modeling pluralistic ignorance. There exist several logics that combine beliefs/knowledge and actions, but which one to chose and the actual modeling, is left for further research to decide.

6.6 Conclusion

Firstly, we have seen that there are many ways of defining pluralistic ignorance, all of which by satisfiable formulas. Therefore, pluralistic ignorance is (seemingly) not a phenomenon that goes against logic. In other words, wrong logical reasoning is not necessarily involved in pluralistic ignorance.

Secondly, the standard definitions of pluralistic ignorance, for instance as a situation where no one believes, but everyone believes that everyone else believes, do not entail that the phenomenon is fragile. Public announcements of the true beliefs of some of the involved agents are not enough to dissolve pluralistic ignorance. Either all the agents need to announce their true beliefs or new information has to come from an outside, trusted source. However, pluralistic ignorance often seems to occur in cases where the agents' beliefs are correlated and in such cases pluralistic ignorance might be increasingly more fragile.

The paper has hinted at a first logic approach to pluralistic ignorance. Some features and problems have been singled out, but the main aim of the paper was to pave the way for further research into logical modeling of social phenomena such as pluralistic ignorance.

Acknowledgment

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Chapter 7

Logical Knowledge Representation of Regulatory Relations in Biomedical Pathways

(co-authored with Sine Zambach)

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Abstract: Knowledge on regulatory relations, in for example regulatory pathways in biology, is used widely in experiment design by biomedical researchers and in systems biology. The knowledge has typically either been represented through simple graphs or through very expressive differential equation simulations of smaller sections of a pathway.

As an alternative, in this work we suggest a knowledge representation of the most basic relations in regulatory processes *regulates*, *positively regulates* and *negatively regulates* in logics based on a semantic analysis. We discuss the usage of these relations in biology and in artificial intelligence for hypothesis development in drug discovery.

Keywords: Formal relations, semantic analysis, biomedical ontologies, knowledge representation, knowledge discovery, applied logic, formal ontologies.

7.1 Introduction

Regulatory networks are used for simple modeling of varying complexity, for example within biology, economy and other fields which apply dynamic systems.

In biomedicine, regulatory networks are widely used to model regulatory pathways, which, in short, are characterized by processes containing gene products and smaller molecules that regulate each other through different mechanisms through different paths. The relations among the building blocks of these networks are typically modeled either very expressively in *e.g.* linked differential equations within the area of physical chemistry as in [45, 88], or very simply in graphs in information systems as in KEGG and Reactome [103, 118].

In this paper, we take another approach and discuss an initial framework for knowledge representation semantically based on logics. This approach is widely used within knowledge representation and we apply it on an abstraction of the biological notion of regulatory pathways. Our focus is on the relations *positively regulates* and *negatively regulates* as well as neutrally *regulates*, which we assume is a super relation of the two others. We call the three relations "regulatory relations" and we use the terms *stimulates* and *inhibits* interchangeably with *positively-* and *negatively regulates*.

The aim of a logical knowledge representation is to capture the formal semantics of the relations. Furthermore, logic implementations offer an opportunity to reason automatically (in a qualitative way) with the goal of obtaining new knowledge. This representation can be utilized in further work on lexical-semantical annotation to be used in information retrieval systems for example. Additionally, the representation can be a part of simulating regulatory networks in biology in a relatively simple manner.

The use of logic in knowledge representation is not a new thing. For instance, the popular tool for semantic web, OWL, has a semantics based on logic. The logic is a variant of a description logic, a family of logics that have been very popular for knowledge representation. Another classical logic for knowledge representation is first-order logic, which description logic can be seen as a fragment of. We will use first-order logic since it is more expressive, but we will also discuss the possibility of using description logic.

In this paper we will first discuss related work on knowledge representation in the biomedical area in Section 7.2. In Section 7.2.2, we present examples of knowledge on biomedical regulatory pathways. To represent the kind of knowledge described by these examples, the domain in question must be ex-

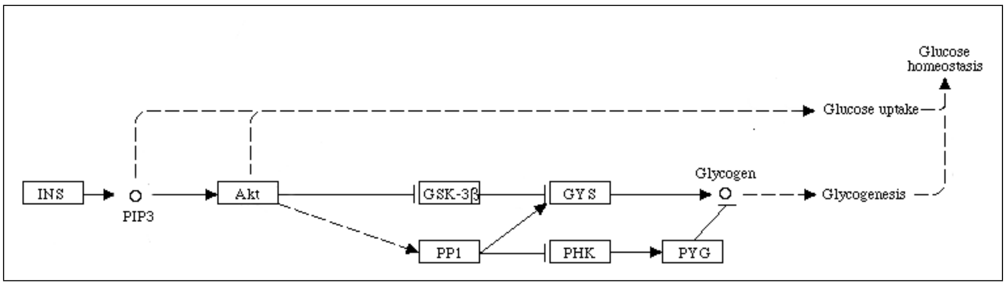

Figure 7.1: A chain of inhibitions and stimulations in the insulin signaling pathway (simplified). The figure is a modified picture from KEGG [103].

posed to a deeper ontological investigation. Furthermore, the clarification of the ontology displays which underlying assumptions are involved in the knowledge we aim to represent. These issues will be discussed in Section 7.3. In Section 7.4, we use First-order Logic to specify a formal semantics of regulatory relations. Next, we analyze the entities involved in regulatory relations, which allows us to clarify regulatory relationships even further. In addition, we discuss the possibilities of representing the relations in Description Logic. Finally, we discuss the biological usage of our formalization of the regulatory networks and further work.

7.2 Related work and examples

7.2.1 Related work

Knowledge representation of biomedical pathways typically spans from simple graph representations among gene products to the more sophisticated linked differential equations, as already mentioned.

Graph representations are mostly informal and constructed to illustrate a regulatory path. In more formal graphs like KEGG [103] (in Figure 7.1), Reactome [118], and MetaCyc [45], regulation among entities like small molecules and gene products are formalized into a database, to which you can have simple queries. For example, in the network of Figure 7.1, the legend tells us that "PP1 activates GYS" (because of the arrow) and "PP1" is a gene product (because of the box around "PP1"), which is information stored in the relatively simple structure of KEGG.

At the other end of the scale, regulatory pathways can be represented using linked differential equations expressed in the formula:

$$dS_i/dt = f_i(S_j, p_k) = f_i^+ + f_i^- \;{}_{k=(1,\ldots,m)}^{i,j=(1,\ldots,n)}. \tag{7.1}$$

For our purpose, the two most important parameters are f_i^+, the sum of the incoming flux (leading to positive regulation), and f_i^-, the sum of the outgoing flux leading to negative regulation [88]. This representation is very expressive and will be difficult if not impossible to implement on the almost 10,000 reactions that are represented in KEGG [45].

Additionally, the problems of acquiring such detailed information and the computationally higher complexity in these models, have lead to suggest less complex models, as for instance in [55]. This simpler model makes it possible to reason qualitatively about existing pathways leading to a rough flux-balance analysis similar to MetaCyc [45] and BioSim [87], which uses Prolog and qualitative constraints.

During the last decade a movement towards formalizing biomedical ontologies and the relations the ontologies contain, has progressed. For example, the widely used Gene Ontology [123, 9] has been ontologically "cleaned up" and initiatives like OBO have provided a framework for the work on formal relations and cooperative ontology modeling [139, 32]. Moreover, concerning properties of relations, the Role Ontology has been developed within the OBO foundry [140]. In the pathway modeling, especially concerning regulation, ontologies and formalized systems like the Gene Regulation Ontology [20], EcoCyc/MetaCyc [45] and Pathway Logics [55] have suggested different approaches to logic representation. Furthermore, in the work on the Gene Regulation Ontology(GRO), regulation is present as a concept. The purpose of that work is to formalize the concepts related to gene regulation used in for example the Gene Ontology [9].

This work introduces First-order Logic in the representation in line with [140], for example.

7.2.2 Examples of regulation

To be able to argue for the ontological assumptions in Section 7.3 and the logical semantic presented in Section 7.4, in this section, we will present examples of the knowledge our simple model is supposed to capture.

The regulates-relations *positively regulates* and *negatively regulates* - are central in for example economics and biochemical pathways. In biomedical pathways, which we will concentrate on in the rest of this paper, it is typical gene products and smaller molecules that interact with each other in complex processes.

A commonly used example of regulation is the insulin response mechanism. Insulin stimulates through a regulatory path, uptake of glucose through cell walls, protein synthesis and glucogenesis, for example. These stimulations occur via for example an activation of PIP3 and the glucogenesis is triggered for example through an activation of the Akt-protein and inhibition of PHK as shown in Figure 7.1. Note here, that most of the regulatory paths are between the biological entities, gene products and small molecules. This typically means that one biological entity regulates the level (by regulating the production/secretion process) or function of another biological entity.

Another example of how regulation in a biological pathway has been represented semi-formally for use in hypothesis testing, is the presentation of the damage response pathway in yeast, which has been investigated and characterized by for example [173]. This work is a usage of a strategy for a longer pathway-traveling concerning gene products that regulate the production of other gene products. The work uses the notion of "deletion-buffering". The meaning of this is: When you remove a transcription factor, X, of the gene product, G, the result is that the production of another protein, P, cannot be regulated anymore by G, if G interacts directly with P. This typically results in an activation of P if G is an inhibitor and an inhibition of P if G is an activator.

There will of course be examples that are non-trivial compared to the above mentioned examples. An example is predicted inhibition in the recently discovered miRNAs, which are small regulating transcripts. miRNAs typically regulate gene products by binding to the mRNA of the gene product. However, often the regulation is just predicted *in silico* by sequence analysis until the experimental data has verified (or falsified) the interaction. This information is difficult to model by a simple regulates-relations since the meaning is rather "predicted to inhibit". We will return to this issue in Section 7.4.3.

7.3 Ontological clarifications

Based on the previous presented examples, we investigate in more detail the ontological aspects of regulatory relations as they present themselves in biomedical research. We clarify which entities are subject to relationships and make a distinction between general concepts or classes and individuals instantiating these classes.

7.3.1 Research practice and granularity

When creating models of knowledge to be used in biomedical hypothesis development, inspiration can be obtained by the practice in the biomedical laboratory. Before the validation of a hypothesis is to be carried out in the laboratory, the precise rates and levels of the involved substances are not necessarily the first thing to consider. Rather we consider a qualitative overview of the processes to be the primary instrument in this stage of discovery. Thus we ignore the precise levels of substances and only care whether their levels are affected positively or negatively, leaving us with a higher level of abstraction (as in [55]).

Likewise, talking about "insulin positively regulates glucose transport", what is really meant is that an amount (or pool) of insulin causes an amount of glucose molecules to be transported. Or, expressed more precisely: When the level of insulin rises, this rise causes a higher frequency of glucose molecules to be transported (through cell walls). This rise is typically directly or indirectly caused by external addition of substance - either by intake of nutrition, medication or even by lack of intake of the necessary nutrition to make the system work properly. Hence "amounts" or "pools" of substances are the basic entities, which are subject to possible relations.

7.3.2 Underlying ontological assumptions - Instances and classes

The former subsection suggests that amounts of molecules rather than single molecules are the central concern of biomedical researchers. In laboratory context, researchers operate with certain amounts or batches of fluids containing multiple molecules. Also, the organs of the human body secrete an amount of molecules for regulating processes. Nothing really happens if the beta cell secretes *one* insulin molecule.

The ontological assumptions we make are virtually in line with the one presented in [140] and [141]. As in [140], we will distinguish between classes and instances. Classes (or concepts or types) refers here to what generally exists, such as insulin, glucose, glucose transport, stem cell, *etc.*. In the following, names of classes will be italicized and begin with a capital letter, for instance *Insulin* and *Glucose_transport*. On the other hand, instances (or individuals, particulars, or tokens) are entities that exist in space and time as instances of a class, such as a particular quantity of insulin or beta cells. We will use $x, y, z, ...$ as variables ranging over arbitrary instances.

The distinction between classes and instances allows us to analyze a natural language expression such as "insulin positively regulates glucose transport" in more details; the sentence does not claim that the class $Insulin$ positively regulates the class $Glucose\_transport$, but that a certain relation between individuals of these classes exists as presented in Section 7.3.1. Thus we assume that relations between the instances are given, for example by experimental evidence in the laboratory, and on the basis of these we define relations among classes or concepts. For instance, "positive regulation" relations may exists among particular amounts of insulin and particular glucose transports, and on the basis of these we define a relation between the classes $Insulin$ and $Glucose\_transport$. This we will do in the next section. Names of relations among individuals will be in bold face, *e.g.* "x **stimulates** y", whereas names of relations among classes will be in italic, *e.g.* "$Insulin\ stimulates\ Glucose\_transport$".

7.4 Analysis of the formal semantic

In this section we provide a thorough semantic analysis of the regulatory relations based on First-order Logic. Finally, we mention the possibilities for an analysis in Description Logic and OWL.

7.4.1 A Logic formalization of regulatory relations

Given the ontological assumptions, we will now discuss the possible relations between classes involved in the knowledge we aim to represent. In the definition of relations among these classes we will use First-order Logic. First-order Logic is used to reason about individuals and their properties, and allows for quantification over these individuals. The language of First-order Logic is built from propositional connectives such as "and" ($\wedge$), "or" ($\vee$), "not" ($\neg$), predicate symbols involved in $P(x)$ or $Q(x, y)$, variables $x, y, z, \ldots$, and quantifiers $\forall$ and $\exists$ (reading "for all" and "there exists"). We have chosen First-order Logic because of its simple reading and generality.[1]

A first example of a formalized relation between classes is the "part of" relation present in many biological ontologies. One can state that $Cell\_membrane$ is "part of" $Cell$, which expresses the fact that every particular cell membrane is the membrane of a particular cell. In other words, "for every cell membrane there exists a cell of which it is part of". Assuming a **part_of** relation between

[1]A commonly used logic for knowledge representation such as Description Logic can be viewed as a fragment of First-order Logic. We will return to the possibilities of giving the semantic in terms of Description Logic later in Section 7.4.4.

individuals, one can define a *part_of* relations among classes C_1 and C_2 in the following way [141]:

$$C_1 \ \mathit{part\_of} \ C_2 \quad \text{iff} \quad \forall x(C_1(x) \rightarrow \exists y(x \ \mathbf{part\_of} \ y \wedge C_2(y))). \tag{7.2}$$

A relation between classes defined this way will for easy reading be called "$\forall\exists$". Generally we define a $\forall\exists$ relation $rel_{\forall\exists}$ between two classes, C_1 and C_2, based on a relation **rel** between individuals, by:

$$C_1 \ rel_{\forall\exists} \ C_2 \quad \text{iff} \quad \forall x(C_1(x) \rightarrow \exists y(x \ \mathbf{rel} \ y \wedge C_2(y))). \tag{7.3}$$

The two classes C_1 and C_2 are also called the *relata* of the relation. Another example of a concrete relation between classes is the one exemplified by the term "enzymes stimulate processes". Even though it may not be visible on the surface, what we have here is an even stronger tie between the two classes than expressed by a $\forall\exists$ relation. The relation between enzymes and process is such that whatever an enzyme stimulates, it is a process. A relation of this kind will be called "$\forall only$". Formally we define the $\forall only$ relation by:

$$C_1 \ rel_{\forall only} \ C_2 \quad \text{iff} \quad \forall x(C_1(x) \rightarrow \forall y(x \ \mathbf{rel} \ y \rightarrow C_2(y))). \tag{7.4}$$

We now consider the case *positively_regulates* as exemplified by a phrase such as "insulin positively regulates glucose transport", which exemplifies the kind of knowledge we aim to represent. As previously discussed in Section 7.3.1, a sentence like this should be read as "for all amounts of insulin and all glucose transports, the insulin can potentially positively regulate the glucose transport". To express a relation like this we introduce the "$\forall\forall$" relation between classes in the following way:

$$C_1 \ rel_{\forall\forall} \ C_2 \quad \text{iff} \quad \forall x(C_1(x) \rightarrow \forall y(C_2(y) \rightarrow x \ \mathbf{rel} \ y)). \tag{7.5}$$

The reason for choosing the $\forall\forall$ relation instead of the $\forall only$ to represent knowledge such as "insulin positively regulates glucose transport", is that Insulin also has the possibility to stimulate other processes such as glycogen production. This possibility is excluded if the knowledge is represented as a $\forall only$ relation.

7.4.2 The relata of regulatory relations

A deeper ontological analysis of the entities involved in regulatory relations reveals that a distinction between continuants and processes has ontological significance. Thus relations between individuals have to be divided into cases depending on whether the individuals are continuants or processes. However, in this section we show that these cases can be reduced to just one case using operators for production and output.

In line with [140] we distinguish between continuants and processes. Continuants are entities that continue to exists over time and may undergo changes, contrary to processes, which are events. Continuants are entities that can change and the changes themselves are processes. We will use $c, c_1, c_2, ...$ to range over continuants and $p, p_1, p_2, ...$ to range over processes. An example of a continuant in our domain could be an amount of insulin, whereas a glucose transport is a process.

In case of regulation, continuants can regulate other continuants or processes, but processes can also regulate other processes or continuants. Thus there seem to be four possible regulatory relations depending on whether the related individuals are continuants or processes. Focusing on the relation "stimulates" there are therefore four possible relations among individuals. We name these $\mathbf{stimulates_{cc}}$, $\mathbf{stimulates_{cp}}$, $\mathbf{stimulates_{pc}}$, and $\mathbf{stimulates_{pp}}$, where for instance the subscript "*cc*" means that it is a relation that can only hold between two continuants. However, introducing a "production of" and a "output of" operator makes us capable of reducing these four relations to only one.

The $\mathit{production\_of}(...)$ operator works on a continuant c by transforming it to the process that is the production of c. Similarly the $\mathit{output\_of}(...)$ operator transforms a process p to the continuant that is the output of p. [2]

With these operators the instance relations $\mathbf{stimulates_{cc}}$, $\mathbf{stimulates_{pc}}$, and $\mathbf{stimulates_{pp}}$ can be reduced to the $\mathbf{stimulates_{cp}}$ relation. These reduction are given by:

c_1 $\mathbf{stimulates_{cc}}$ c_2	reduces to	c_1 $\mathbf{stimulates_{cp}}$ $\mathit{production\_of}(c_2)$
p $\mathbf{stimulates_{pc}}$ c	reduces to	$\mathit{output\_of}(p)$ $\mathbf{stimulates_{cp}}$ $\mathit{production\_of}(c)$
p_1 $\mathbf{stimulates_{pp}}$ p_2	reduces to	$\mathit{output\_of}(p_1)$ $\mathbf{stimulates_{cp}}$ p_2

These reductions reflect how the relations are used as verbs in sentences in biological texts, for example: "Insulin $\mathbf{stimulates_{cc}}$ glycogen", "insulin

[2] However, the biological domian is not always this simple. Some processes may of course have several outputs. In larger regulatory pathways processes may regulate other processes, but this is always through outputs of the first process. These outputs can be unknown or it can be unknown which of the outputs that actually regulates the second process. This will be touched upon in the discussion.

$\mathbf{stimulates_{cp}}$ the glycogenesis", and "insulin $\mathbf{stimulates_{cp}}$ the production of glycogen (through the glygonenesis)" where the process glycogenese is equal to "the production of glycogen". Likewise you can formulate the sentences: "beta cell secretion $\mathbf{stimulates_{pp}}$ glycogenese" that can be reduced to "output of beta cell secretion $\mathbf{stimulates_{cp}}$ production of glycogen", where the output of beta cell secretion is insulin.

7.4.3 Modal, temporal and spatial aspect of regulatory relations

In the reading of "Insulin stimulates glucose transport" as "for all amounts of insulin and all glucose transports, the insulin can potentially positively regulate the glucose transport" the term "*can potentially*" plays a considerable role. It is a vague modal term, and in this section we will attempt to make it more precise.

When representing "Insulin stimulates glucose transport" as

$$\forall x(Insulin(x) \rightarrow \forall y(Glucose\_transport(y) \rightarrow x\ \mathbf{stimulates_{cp}}\ y)),$$

the term "can potentially" is implicit in the relation "$\mathbf{stimulates_{cp}}$". In other words "x $\mathbf{stimulates_{cp}}$ y" is read as "x can potentially stimulate y". This seems sensible since "Insulin stimulates glucose transport" does not express that all amounts of insulin *actually* stimulates all glucose transports, but that they potentially can. Using modal logic formalisms [31] we can replace "x $\mathbf{stimulates_{cp}}$ y" by "$\Diamond(x\ \mathbf{stim}\ y)$", where the "$\Diamond$" is a modal operator representing "potentiality" and "**stim** " is the primitive name of a relation of actual stimulation. Thus "c **stim** p" means that the continuant c actively stimulating the process p. A semantics for this elaborated formula can then be given in the framework of first-order modal logic [40].

The potentiality represented by the $\Diamond$ operator can, however, be analyzed even further in the case of the stimulation relation. There are two kinds of vagueness involved in the "can potentially stimulate" expressed in "$\Diamond$(...**stim**...)". The first one is due the fact that stimulation only takes place if the substance is actively participating in the process. If the process and substance are separated in space and time, stimulation can of course not take place. The relation of a continuant taking actively part in a process at a given time, is a basic relation and in [140] it is assumed as a primitive relation. Using their notation "p **has_agent** c **at** t" expresses that the continuant c is causally active in the process p at time t. Together with the **stim** relation,

"c $\mathbf{stimulates_{cp}}$ p" can thus be expanded as:

$$\forall t(p\ \mathbf{has\_agent}\ c\ \mathbf{at}\ t \rightarrow c\ \mathbf{stim}\ p),$$

hence the $\Diamond$ operator is unnecessary.

The other vagueness involved in "can potentially stimulate" is due to the fact that other substances may interfere with the process, in which case a stimulation may not take place. Thus, in the above formula, x **stim** y should be read as "if no other substances interfere an actual stimulation between x and y takes place". If we use the formalization with the *Diamond* operator, we can include this *ceteris paribus* clause in the reading of the $\Diamond$.

In non-trivial examples as the predicted regulation by miRNAs as described in Section 7.2.2 there is an additional vagueness. This is due to the stimulation only being *predicted*. For example, *miRNA* $\mathbf{stimulates_{cc}}$ c (predicted *in silico*), and c $\mathbf{stimulates_{cp}}$ p should lead to a weaker inference between *miRNA* and p than if the *miRNA* was experimentally shown to stimulate c.

7.4.4 Description Logic representation of class relations

In Section 7.4.1 we used First-order Logic to present a formal semantics for regulatory relations. This was motivated by the easy reading of First-order formulas that also made the difference between the class relations $rel_{\forall only}$ and $rel_{\forall\forall}$ visible. In this section we discuss the possibility of defining the relations in Description Logic.

Description Logic is a family of logics widely used for knowledge representation, and in several of the logics reasoning can be done in low complexity contrary to First-order Logic which is undecidable (for more on the complexity of Description Logic see chapter 3 in [12]). Furthermore, Description Logic is also implemented in several modern tools such as OWL [76]. Proteg-OWL is a popular language for knowledge representation and it can implement most flavours of Description Logics although the OWL-full version of OWL.1 is undecidable.

The two class relations $rel_{\forall\exists}$ and $rel_{\forall only}$(defined in (7.3) and (7.4)) can easily be formalized in Description Logic by:

$$\begin{array}{lll} C_1\ rel_{\forall\exists}\ C_2 & \text{iff} & C_1 \sqsubseteq \exists\mathbf{rel}.C_2, \\ C_1\ rel_{\forall only}\ C_2 & \text{iff} & C_1 \sqsubseteq \forall\mathbf{rel}.C_2. \end{array}$$

However, the class relation $rel_{\forall\forall}$ is not expressible in a majority of Description Logics. Although in very expressible Description Logics including full concept negation and role negation [116, 114], the $rel_{\forall\forall}$ relation can be formalized by:

$$C_1\ rel_{\forall\forall}\ C_2 \quad \text{iff} \quad C_1 \sqsubseteq \forall(\neg\mathbf{rel}).\neg C_2.$$

Alternatively a new operator in line with the "$\exists$**rel**" and "$\forall$**rel**" operators could be added to a Description Logic. Such an operator has already been added to similar modal logics and goes under the name "the window operator". However, a minimal Description Logic with this operator appears not to have been investigated. Thus, there is still work left to be done in the field of Description Logic before knowledge on regulatory relations can be optimally represented.

7.5 Discussion

Based on an analysis of the biomedical examples and our declaration of the ontological assumption, we have suggested that the correct formalizations of *positively* and *negatively regulates* in First-order Logic are represented by the formula $\forall x(C_1(x) \rightarrow \forall y(C_2(y) \rightarrow x\ \mathbf{rel}\ y))$. A description of the relata, the First-order formulas, and examples of *regulates*, *positively_regulates* and *negatively_regulates* are displayed in Table 7.1.

Thus, our contribution to the field of knowledge representation and biomedical informatics is a logical analysis and representation of regulatory relations. One of the main advantages of modeling knowledge in a formal framework as logic is that it makes entire knowledge bases available for consistency checks and allows for the use of reasoning tools to gain new knowledge. This is particularly useful in for instance artificial intelligence and information retrieval.

In relation to the related work of Section 7.2, this formalization is in the middle of a complexity scale. It is not as expressive as the linked differential equations [88], but much better suited for automatic reasoning than simple graphs [103]. In expressivity and tractability it is close to work like [55, 45]. However, this work provide a semantic and uses First-order Logic formalization, which provides more information to the relations than the before mentioned due to the quantifications.

7.5.1 Applications in the biomedical domain

From a biological point of view, the main purpose of our formalization of the regulatory relations is to assist knowledge discovery, hypothesis development,

Table 7.1: Definitions of three regulatory relations. They are expressed as class-level relations in a format similar to that of OBO Relation Ontology [140]. *Relation and relata* capture the representation in e.g. KEGG, *Definitions* displays the First-order Logic formalizations, and *Examples* contributes with examples taken from pubmed-abstracts [132].

A. Regulates

Relations and relata	C_1 $regulates_{\forall\forall}$ $production\_of(C_2)$; C_1 and C_2 are continuants.
Definitions	$\forall x(C_1(x) \rightarrow \forall y(production\_of(C_2(y)) \rightarrow$ x **regulates** $y))$.
Examples	...nitric oxide pathway regulates pulmonary vascular tone... ...non-histone chromosomal proteins may modify gene expression ... ...CREB regulates cyclic AMP-dependent gene...

B. Positively Regulates

Relations and relata	C_1 $positively\_regulates_{\forall\forall}$ $production\_of(C_2)$; C_1 and C_2 are continuants.
Definitions	$\forall x(C_1(x) \rightarrow \forall y(production\_of(C_2(y)) \rightarrow$ x **positively_regulates** $y))$.
Examples	...IPA stimulates insulin release... ...Ca(2+) influx stimulates exocytosis of secretory granules... ...MMP-7 activates the epidermal growth factor...

C. Negatively Regulates

Relations and relata	C_1 $negatively\_regulates_{\forall\forall}$ $production\_of(C_2)$; C_1 and C_2 are continuants.
Definitions	$\forall x(C_1(x) \rightarrow \forall y(production\_of(C_2(y)) \rightarrow$ x **negatively_regulates** $y))$.
Examples	...GLP-1inhibits glucagon release... ...lithium inhibits the enzyme glycogen synthase kinase-3... ...RSBX negatively regulates an extension of the RSBV-RSBW pathway... ...insulin secretion from the β-cell to reduce IRI responses...

and, in a broader perspective, lexical resource integration of the semantics of the words representing the relations.

7.5.1.1 Inferences and reasoning rules.

Proposing rules for reasoning in a logical framework allows us to obtain new knowledge from an existing knowledge base.

First, we have the inferences that are given from the semantics of the First-order Logic. For the relation *regulates* defined as in equation (7.5) we will have the following inferences:

$$is\_a \circ regulates \rightarrow regulates$$
$$regulates \circ is\_a^{-1} \rightarrow regulates$$

The $\circ$-operator is a common notation for composition of relation and the *is_a* relation is interpreted as the subset relation.

Additionally, you can create domain- or application-specific reasoning rules depending on the amount of knowledge you want from your system. In implementing artificial intelligent systems in biomedical informatics several reasoning rules have been suggested [175, 123].

A list of proposed rules can be found in [175][3] and in [123], and an example of one of these is:

$$negatively\_regulates \circ negatively\_regulates \rightarrow positively\_regulates$$

From the reasoning rules we can deduce additional relationships from existing ones, and we can make inferences such as: "if insulin stimulates glucose transport and if the glucose transport inhibits glyconeogenesis, then insulin inhibits glyconeogenesis". Thus, if you want to find novel gene products and molecules that regulate a given process or a given molecule in a certain way, you can use reasoning rules to predict such. Another perspective of this automated reasoning is the prediction of the side effects of a drug or extra molecule functions.

Furthermore, you can potentially place a new unfamiliar molecule correctly in a regulatory pathway due to its regulatory properties. These functions can be an advantage in drug discovery, identification of adverse effects and in knowledge expansion for more fundamental research purposes. These are just

[3]The rules of [175] are meant to hold throughout the biomedical domain, however they may not always reflect reality. Instead, they shoulde be viewed as rules of thumb, used to infer possible new knowledge in drug discovery and hypothesis development.

some of the many advantages a logic based knowledge representation, as the one presented here, provides when fully implemented.

7.5.1.2 Towards implementation of a prototype.

The most straightforward evaluation would obviously be an implementation of a prototype system for information retrieval and/or for hypothesis generation that would use the suggested formalisation. A comparison with similar systems not using the same formal representation of regulatory relations, would then make the contribution of the semantical representation clear.

To illustrate the effects and properties of the relations we have constructed a small example in the logical programming language Prolog. We have implemented a small part of the KEGG database from Figure 7.1 containing 21 classes and the relations: *is a*, *stimulates* and *inhibits*. Besides the relations in the figure, a small taxonomy is created such that we are able to separate continuants such as (*small_molecule* and *gene_product*) and processes in correspondence to the way KEGG names the entities.

The toy-implementation can be used to infer fundamental inheritances in taxonomies of classes (ontology consisting of pure ISA-relations) as mentioned in the former subsection and can be downloaded and tested from: *www.ruc.dk/sz/Regrel*. Further work needs to be done to prove that the semantics of the implemented relations are actually equal to the semantics we have suggested in this paper.

Another possibility is to implement the system in DL using the suggestion in Section 7.4.4. However, this will require both full role-negation and full concept negation and the tractability of this is to be investigated further.

7.5.1.3 Ontological aspects of regulation.

In Section 7.4.2 we made a distinction between continuants and processes, leading us to a characterization of 4 different basic relations among individuals. For stimulation these where the relations $\mathbf{stimulates_{cc}}$, $\mathbf{stimulates_{cp}}$, $\mathbf{stimulates_{pc}}$, and $\mathbf{stimulates_{pp}}$, which we further reduced to the single relation $\mathbf{stimulates_{cp}}$. However, one may argue that the relations $\mathbf{stimulates_{pc}}$ and $\mathbf{stimulates_{pp}}$ are not genuine relations in the first place. From a strict ontological point of view processes never stimulate other processes or continuants directly, but always through their outputs.

An example of this is glycogenesis[4]. Glycogen is an output of this process, but other outputs occur as well, for example uridine diphosphate (UDP), whose effects might be different that glycogen. Thus, when we have a statement that the glycogenesis stimulates glucose homeostasis, we cannot be certain whether glycogen or UDP or both are the actors unless this is stated explicitly. Nevertheless it is either glycogen or UDP (or both) that stimulate homeostasis, and not actually glycogenesis.

We recognize that it is a debatable issue whether processes can stimulate other processes or continuants. There seems to be evidence, however, that it is important to investigate the ontological aspects of stimulation further. Whether stimulation is among continuants or processes seems to have consequences for the inference of new knowledge, and thus the distinction should be recognized. In simple knowledge representations by graphs like in the KEGG database such observations are not accounted for. Such knowledge bases have the potential to get this representation integrated automaticaly when a semantics is agreed upon.

7.5.2 Future work

With our discussions and examples we have revealed that several occurrences of regulatory relationships are characterised by vagueness or fuzziness. However, one could take it one step further by taking the characteristic fuzziness seriously and apply fuzzy logics or other logics of uncertainty to model this aspect of the regulatory relationships.

As mentioned in Section 7.4.4 the logical framework of Description Logic is still not fully developed for representing the formal semantics of regulatory relations as we defined them in this paper. This is also a direction of future work that may tell us something about how efficient we can do automatic reasoning about regulatory relations in practice. Furthermore it may also reveal how the relations can be incorporated into for instance OWL.

Staying within the field of logic, we mentioned several possible class relations in Section 7.4.1 and a deeper analysis of all possible class relations and there properties would also be interesting future work.

We finally suggest that the formal semantic analysis presented in Section 7.4 can be used to define frame semantics for the verbs and verb phrases that express regulatory relations. The definition of the frame semantics is sim-

[4]Note that *glycogenesis* and the above mentioned *glyco**neo**genesis* are two different processes.

ilar to the one we find in the work done in BioFrameNet [53] on other verbs. The attempt of defining frame semantics may well result in an enrichment of semantically annotated data and for instance be applied to semantic querying [6].

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Chapter 8

Conclusion

Abstract: The chapters of this dissertation deals with many different issues and it may be hard to see how they are connected as well as what the main contributions of this dissertation are. Thus, the aim of this final chapter is to draw attention to the main themes of this dissertation and highlight the main contributions it makes.

The introduction presented the logics central to this dissertation, namely modal logic, hybrid logic, epistemic logic, dynamic epistemic logic, and many-valued logic. The aim of the introduction was to further promote the view of logic as a formal toolbox useful for conceptual modeling of important notions such as knowledge, information, and beliefs. However, the main concern of this dissertation has been to further develop these formal tools. More specifically the majority of attention has been given to the technical issues involved in expanding existing many-valued, hybrid, and dynamic epistemic logics and developing new proof theory for these. Still, the last two chapters have been devoted to logic-based modeling of concepts from social epistemology and biomedical informatics. Generally, the dissertation can be seen as being concerned with the logic toolbox useful for modeling knowledge and information. Additionally, the dissertation deals with several more specific themes where several important contributions are made. These themes and contributions will now be discussed.

Hybrid logic plays an important role throughout the dissertation as it is involved in the chapters 2, 3, 4, and 5. Especially the proof theory of hybrid logic is a central theme and the main technical results in this dissertations involves extensions of the proof theory of hybrid logic to new logics in the chapters 2, 4, and 5. From a semantic perspective, this dissertation shows that hybrid logic extensions are possible beyond standard modal logic as hybrid logic versions of many-valued modal logic are introduced in the chapters 2 and 3 and a hybrid logic version of public announcement logic is introduced in Chapter 4. One interesting insight regarding the semantics of hybrid logic, that this dissertation provides, is the fact that the equivalences between the formulas $@_i\varphi$, $E(i \wedge \varphi)$, and $A(i \rightarrow \varphi)$ are something particular to standard hybrid logic. In many-valued hybrid logic or hybrid public announcement logic these equivalences may fail. More generally, chapters 3 and 4 shows that the semantics of nominals and satisfaction operators are not always a canonical choice when developing more advanced hybrid logics.

Regarding the proof theory of hybrid logic, a Hilbert style axiom system is shown to be extendable to hybrid public announcement logic in Chapter 4 and the proof theoretic advantage of general completeness for logics obtained by adding pure formulas as axioms is preserved (at least in some sense[1]). The general completeness result can be seen as a first step in an investigation of public announcement logics for other kinds of frame classes than the usual **K** and **S5**. Public announcement logics for other kinds of frame classes is an uninvestigated topic that hopefully will be given more attention in the future.

Furthermore, a technique used to show that tableau systems for hybrid logics give rise to decision procedures is shown to be very general as it is extendable to both many-valued hybrid logic (Chapter 2), hybrid public announcement logic (Chapter 5), and dynamic epistemic logic (Chapter 5). This insight may lead to a general way of constructing terminating tableau systems for new logics.

Proof theory of dynamic epistemic logic is another central theme in this dissertation. As discussed in Section 1.2.3 of the introduction the proof theory of dynamic epistemic logic is especially underdeveloped. However, the importance of proof theory for dynamic epistemic logic is not to be underestimated and hopefully it will lead to more research in the future. The chapters 4 and 5 contribute to the proof theory of dynamic epistemic logic by providing a complete Hilbert style axiomatization of hybrid public announcement logic with pure axioms and by providing terminating tableau systems for hybrid public announcement logic and dynamic epistemic logic with action models. Furthermore, the discussion in Chapter 4 about adding distributed knowledge using pure formulas shows that there might be more to Hilbert style axiomatizations of dynamic epistemic logics than is generally believed in the community. For

[1] Recall the discussion from Chapter 4 regarding adding distributed knowledge using pure formulas.

instance, the combination of axiom systems of public announcement logic with axiom systems for the logic **KD45** can be tricky.

In addition to contributions to technical issues in modal logics, this dissertation also makes some contribution to issues concerning knowledge, information, and beliefs as they are discussed in social epistemology. On topic that social epistemology deals with is judgment aggregation [75], which in Section 1.3.3 is discussed in the light of many-valued logics. Another topic is pluralistic ignorance [91], which is modeled in Chapter 6 using a dynamic epistemic logic. In Chapter 6 it is shown, by a formalizations in a dynamic epistemic logic, that the phenomenon of pluralistic ignorance in not logically inconsistent and can be a robust phenomenon on the standard account. It is not a final logic-based analysis of pluralistic ignorance by a first step in modeling social phenomena from an information perspective using logic. Section 1.3.3 does not contain any results at all, but it does suggest that studying judgment aggregation problems together with decision problems may lead to interesting research in the future and that many-valued logics may be a way to go.

Finally, Chapter 7 aims to show that when designing knowledge representation formalisms, logic is a useful tool, but it requires several ontological clarifications before it can be used properly. However, such clarifications are important when using the logic to infer new knowledge.

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