ROSKILDE UNIVERSITETSCENTER
Faggruppen for filosofi og videnskabsteori

ROSKILDE UNIVERSITY
Section for philosophy and science studies

Leonardo Fibonacci and *abbaco* culture
a proposal to invert the roles

Jens Høyrup

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In memoriam David Fowler,
gentle friend and wise colleague

The received view

As long as the existence of the late medieval and Renaissance Italian *abbaco* tradition has been recognized, it has been taken for granted by almost everybody that it had to descend from Leonardo Fibonacci's writings, at most with more or less marginal additions. In particular, this has been the repeated view of those scholars who know the tradition most intimately and who have made it known to the rest of the world

The latest phrasing of the view may be that of Elisabetta Ulivi [2002: 10], according to whom the *libri d'abbaco* "were written in the vernaculars of the various regions, often in Tuscan vernacular, taking as their models the two important works of Leonardo Pisano, the *Liber abaci* and the *Practica geometriae*".[1]

Even stronger is Warren Van Egmond's statement [1980: 7] that all *abbaco* writings "can be regarded as [...] direct descendants of Leonardo's book". As regards *abbaco algebra* in particular, the same author asserts that this "tradition is logically a continuation of the work of Leonardo Pisano" though not explaining which logic should be involved [Van Egmond 1988: 128].

In [1985: 28], Raffaella Franci and Laura Toti Rigatelli stated similarly that "the abacus schools had risen to vulgarize, among the merchants, Leonardo's mathematical works".[2] As regards the algebra contained in some of the treatises, however, Franci and Toti Rigatelli already mitigated the claim just quoted in the same article by the observation (p. 45) that

> in Florence, in the 14th century, at least two algebraic traditions coexisted. One of

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[1] As everywhere in the following where no translator is identified, I am responsible for the translation.

[2] More recently, Franci [forthcoming] has downplayed the importance of the *Liber abbaci* significantly, suggesting instead that the inspiration was derived from that lost *liber minoris guise*, "book in a smaller manner", which Fibonacci says to have written [ed. Boncompagni 1857: 154]. I shall return to my reasons for finding this implausible in note 10, cf. also note 23.

> them was inspired by Leonardo of Pisa and was improved by Biagio the Old and Antonio de' Mazzinghi, the other, the beginning of which is unknown until now, has Gerardi as its first exponent.

In [2002: 82], Franci sharpened this dissent from the prevailing view, suggesting that the "authors [of fourteenth-century *abbaco* algebra] may have had access to Arabic sources different from those used by Leonardo". Partial divergence from the conventional wisdom is also expressed by Enrico Giusti [2002: 115], according to whom some of the *abbaco* writings

> were genuine and proper vernacular versions of [Fibonacci's] works, made easier by elimination of the most abstract and theoretical parts; in other cases the author limits himself to dig in the mine of examples and problems from the *Liber abaci*, in order to find material he could insert in his own treatise.

Similar partial divergence was expressed by Gino Arrighi already in [1987: 10], when he suspected Paolo Gherardi's *Libro di ragioni* (also referrred to by Franci and Toti Rigatelli) and another treatise which he ascribed to the same author to be either re-elaborations or translations of French writings; on the other hand he stated (p. 5) that these treatises are the only witnesses we have of important mathematical exchanges between Italy and France (i.e., the Provençal area[3]).

Before the autonomous existence of the *abbaco* tradition was recognized, it was even more obvious to those few who did work on *abbaco* material that it belonged within a current leading from Fibonacci to Luca Pacioli, Tartaglia and Cardano. One clear enunciation is due to Louis Karpinski [1929: 177], who ends his description of Jacopo da Firenze's *Tractatus algorismi* from 1307 with the observations that the

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[3] Politically, Montpellier was only definitively integrated in the French Kingdom in 1349 (which did not in itself make it culturally French), after having been bought from the Aragon-Majorcan king; Avignon and the surrounding Comtat Venaissin were only absorbed by France in 1791. Thirteenth-century practical arithmetic from France proper, as known from the last part of the *Pratike de geometrie* [ed. Victor 1979] was very different in character from what we know from Jacopo da Firenze's *Tractatus algorismi* and Paolo Gherardi *Libro di ragioni* (both written in Montpellier, in 1307 and 1328, respectively), and also from a *Trattato di tutta l'arte dell'abacho* (Rome, Biblioteca dell'Accademia Nazionale dei Lincei, Cors. 1875, with parallel manuscripts) written in Avignon in the 1330s (see [Cassinet 2001]; the ascription of the latter treatise to Paolo dell'Abbaco, e.g. in [Van Egmond 1977], is apparently based solely on a probably ill-founded guess by a fifteenth-century owner of one of the manuscripts).

treatise by Jacob of Florence, like the similar arithmetic of Calandri, marks little advance on the arithmetic and algebra of Leonard of Pisa. The work indicates the type of problems which continued current in Italy during the thirteenth to the fifteenth and even sixteenth centuries, stimulating abler students than this Jacob to researches which bore fruit in the sixteenth century in the achievements of Scipione del Ferro, Ferrari, Tartaglia, Cardan and Bombelli.

One reason for the persistence of this belief (which, as I shall argue, is largely illusory) is probably the *principle of the great book*, to which scholars are prone to fall victims: the belief that everything in a book, if not an innovation, must be derived from *a famous book* that is *known to us* – known at least by name and fame if no longer extant.

In a way, this principle can be seen as a sound application of Occam's razor: explanatory entities in the shape of wonderful secret traditions that have left no traces should not be multiplied without necessity. But if applied without attention to the copious evidence that is offered by less famous sources, without regard for the details of the material and without recognition of the fact that this extant material may contain more holes than cheese, then it can at best be compared to Kepler's explanation of planetary movements by means of magnetism, the only force acting at a distance he knew.

However, the creed of modern scholars is only half of the explanation. Early sources also seem to suggest a key role for Fibonacci. In the *Ars magna*, Cardano [1663: 222] tells that algebra took its beginning with al-Khwārizmī and was copiously developed by Fibonacci; much later, as he further relates, three new derivative chapters were added by an unknown author, being put together with the others by Luca Pacioli.

But we may go even further back. *Abbaco* writers of the mature tradition, if referring at all to intellectual ancestors (which they do not do too often), tend to mention Fibonacci, perhaps together with more recent *maestri d'abbaco*. Moreover, already (what is likely to be) the oldest extant *abbaco* treatise presents itself as a *Livero de l'abbecho* "secondo la oppenione de maiestro Leonardo de la chasa degli figluogle Bonaçie da Pisa" [ed. Arrighi 1989: 9], an "*Abbacus book* according to the opinion of master Leonardo Fibonacci". This seems to leave little doubt that Fibonacci *was* indeed a founding father of *abbaco* mathematics, if not *the* father.[4]

The Umbrian evidence

This earliest extant *libro d'abbaco* (Florence, Riccardiana, MS 2404, fols 1^r–136^v) appears from internal evidence to have been written in c. 1288–1290 in Umbria.[5] Whoever starts reading attentively beyond the introductory lines that were just quoted will discover that it contains material that is definitely not from Fibonacci; further on he will also find indubitable borrowings from the *Liber abbaci*. Is this then really a "genuine and proper vernacular version" of Fibonacci's work, made easier by elimination of the most abstract and theoretical parts? Or has the author limited himself "to dig in the mine of examples and problems from the *Liber abbaci*, in order to find material he could insert in his own treatise"? Or is the character of the treatise more fittingly described in some third way?

In order to find out we shall need a close examination of the contents of the treatise. Before that, however, a few words about Fibonacci's way to write mixed numbers and composite fractions will serve.

In the writing of mixed numbers Fibonacci follows what he is likely to have been taught by those teachers in Bejaïa in present-day Algeria with whom he spent "some days studying the abbacus" during his boyhood, as he explains in the preface to the *Liber abbaci* [ed. Boncompagni 1857: 1] – that is, writing them with the fraction to the left, preceding (in our view) the integer part, 182 and a half appearing hence as $\frac{1}{2}182$.

Fibonacci also explains and makes use of several types of composite fractions.

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4 In the interest of moral balance I shall cite my own [2000: 56] as an example of a scholar taken in by this title and the identification of some indubitable borrowings.

5 The actual date may be slightly later, cf. note ?. It should be observed that improved understanding of the coin list contained in the "Columbia Algorism" (Columbia University, MS X 511 A13, [ed. Vogel 1977]) due to Lucia Travaini [2003: 88–92] shows that at least this list (which is not annexed to the text but integrated) was made in the years between 1278 and 1282. The manuscript has habitually been ascribed to the first half of the fourteenth century and is indeed a copy [Vogel 1977: 6], and the Umbrian *Livero dell'Abbecho* is therefore still likely to be the earliest *extant* abbacus manuscript; but the Columbia Algorism now seems to be a copy of the earliest earliest *abbaco* treatise *we know about*, written in or in the vicinity of Cortona.

One renders the "ascending continued fractions" that were commonly used in Arabic arithmetic – $\frac{1\ 5\ 7}{2\ 6\ 10}$ [ed. Boncompagni 1857: 24] thus stands for $\frac{7}{10}$ plus $\frac{5}{6}$ of $\frac{1}{10}$ plus $\frac{1}{2}$ of $\frac{1}{6}$ of $\frac{1}{10}$.[6] Another one stands for stepwise extended products of fractions – $\circ\frac{2\ 4\ 6\ 8}{3\ 5\ 7\ 9}$ thus for $\frac{8}{9}+\frac{6}{7}\cdot\frac{8}{9}+\frac{4}{5}\cdot\frac{6}{7}\cdot\frac{8}{9}+\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdot\frac{8}{9}$(there are still others, but we shall not need them in the following).

Let us then return to the Umbrian *abbaco*. It consists of 31 chapters:
Ch. 1, *de le regole de le tre chose*.
Ch. 2, *de le chose che se vendono a centonaio*.
Ch. 3, *de le regole de pepe che senno*.
Ch. 4, *de le regole degle drappe che se vendono a channa e a br*.
Ch. 5, *de regole de chanbio*.
Ch. 6, *de baracta de monete e denari*.
Ch. 7, *de le regole de marche Tresce* [from Troyes / JH] *e de svariate ragione de lib*.
Ch. 8, *da sapere quante d. de chantra e charrubbe e grana è l'onzia*.
Ch. 9, *de conparare bolçone a numero de denare ed a peso de libr*.
Ch. 10, *de regole de consolare ed alegare monete*.
Ch. 11, *de svariate regole che s'apartengono al consolare de le monete*.
Ch. 12, *de regole de merto o vero d'usura*.
Ch. 13, *de regole che s'apartengono a quille de la usura*.
Ch. 14, *de regole de saldare ragione*.
Ch. 15, *de svariate regole de conpagnie*.
Ch. 16, *de chonpare de chavagle*.
Ch. 17, *de huomene che demandavano d. l'uno a l'altro*.

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[6] That is, $\frac{7+\frac{5+\frac{1}{2}}{6}}{10}$. The name should not mislead, evidently the only link of such expressions with continued fractions proper is graphical. Their spoken form is used routinely in Arabic (and other Semitic languages, see [Høyrup 1990]); Fibonacci's notation coincides with the one that is used in al-Qalasādī's *Kašf* [ed. Souissi 1988: Ar. 67], and there is thus no doubt that Fibonacci has borrowed it from the Maghreb school, even though we may doubt that it belonged to what could be covered in school in a couple of days.

Ch. 18, *de huomene che trovaro borsce.*
Ch. 19, *de huomene che cholsero denare emsiememente.*
Ch. 20, *de regole de prochacio overo de viage.*
Ch. 21, *de huomene ch'andaro a guadagnare agl merchate.*
Ch. 22, *de choppe e del suo fondo.*
Ch. 23, *d'arbore o vogle de legne.*
Ch. 24, *de vasa.*
Ch. 25, *de huomene che vonno per via chumunalemente ensieme.*
Ch. 26, *de huomene che portaro margarite a vendere em Gostantinuopole.*
Ch. 27, *de tine e de botte cho' n'esce el vino per gle foramene cho sonno el fondo.*
Ch. 28, *d'uno che manda el figlo en Alixandria.*
Ch. 29, *d'uno lavoratore che lavorava enn una uopra.*
Ch. 30, *de huomene ch'andano l'uno po' l'altro.*
Ch. 31, *de regole per molte guise forte e ligiere de molte contintione.*

A detailed description of the contents of each chapter is given in the appendix. Here we may sum up some of the general observations that can be synthesized from these descriptions.

Chapters 1–9 and 13–15 borrow nothing from Fibonacci. They all treat of such basic matters as would be of real use for the students of an abbacus school: the rule of three; shortcuts allowed by the metrological system;[7] shrinkage due to the refining of spices; exchange of coin against coin, bullion or goods; metrology, refining and evaluation of bullion; simple interest; and partnerships. Not a single problem in these chapters comes from Fibonacci. Chapters 10 and 12 start by problems of direct relevance for daily commercial life, similarly independent of Fibonacci. The remainder of these two chapters – a collection of reverse alloying problems and one containing problems about giving a loan in a house which the creditor rents – is borrowed from the *Liber abbaci*.[8] So

[7] Rules of the type "if something is sold at *p libre* for a hundred units, then the price of one unit is $2\frac{2}{5}\cdot p$ *denari*" (1 *libra* = 20 *soldi* = 240 *denari*).

[8] Since readers are more likely to have access to Laurence Sigler's translation than to Boncompagni's edition, it may be appropriate to point out that the translation misunderstands the original on this point – see [Sigler (trans.) 2002: 384] confronted with [Boncompagni (ed.) 1857: 267].

is almost everything contained in chapters 11 and 16–30: in part artificially complex problems in commercial apparel, in part variants of well-known «recreational» problem types. Only chapters 22–23, teaching the method of the single false position, is likely to have been useful; the rest might be regarded as brain gymnastics – had it not been for counterevidence to be presented imminently. Chapter 31 is a mixed collection of mainly recreational problems, some from the *Liber abbaci*, others not. Some of the latter are simpler versions of Fibonacci problems that appear in the preceding chapters, reflecting the familiar fact that Fibonacci borrowed amply from a fund of problems that circulated in numerous versions, and suggesting that Fibonacci may have had a predilection for the more difficult of these – those where the need for a mathematical explanation might be urgently felt.

Until near the end of chapter 2, mixed numbers are written with the integer to the left ($3\frac{1}{4}$, etc.). Then suddenly the writing shifts to Fibonacci's system, the fraction being written to the left of the integral part ($\frac{2}{13}1$, etc.); this system remains in vigour until the end.[9] In consequence we see numerous writings of concrete numbers in the awkward style "d. $\frac{17}{49}7$ de denaio", "*denari* $\frac{17}{49}7$ of *denaro*" (meaning $7\frac{17}{49}$ *denari*) – but exclusively in problems that are not taken over from the *Liber abbaci*. A few slips shows that the author has copied rather

[9] So far, only Fibonacci and no other preceding Latin or European-vernacular source is known from where the compiler could have taken his inspiration for this system, and it has thus seemed obvious that Fibonacci *was* the inspiration. But both the Florence manuscript of Jacopo da Firenze's *Tractatus algorismi* (Ricc. 2236) and the two manuscripts of the *Trattato di tutta l'arte dell'abacho* I have inspected (cf. note 3) contain multiplications arranged from right to left in tables; in Ricc. 2236, some of these tables contain products of mixed numbers, writing these with the fractional part to the left. The Columbia Algorism, on its part (see note ?), contains occasional notations for ascending continued fractions, not wholly in Fibonacci's style and written at times from right to left, at times from left to right – $\frac{1}{4}\frac{1}{2}$ meaning $\frac{1+\frac{1}{4}}{2} = {}^{5}/_{8}$ in one place, $\frac{1+\frac{1}{2}}{4} = {}^{3}/_{8}$ in another [Vogel 1977: 13]. Independent influence from Maghreb notations thus turning up in various places, it is not totally excluded that the Umbrian compiler had adopted his "Arabic" ways from a non-Fibonacci source.

faithfully from sources using the straightforward style familiar from other *abbaco* writings (and also used in chapter 1, before the inversion of the writing of mixed numbers), “d. 7, $\frac{17}{49}$ de denaio”, “*denari* 7, $\frac{17}{49}$ of a *denaro*”, trying only to impose on the material the Fibonacci style.[10]

Since the compiler adopts from Fibonacci almost exclusively the intricate matters, he has borrowed numerous problems making use of Fibonacci’s notations for composite fractions. It turns out, however, that he does not understand them. For instance, he reads (to quote one example among many) Fibonacci’s $\frac{33}{53}\frac{6}{53}\frac{42}{53}\frac{46}{53}$ [ed. Boncompagni 1857: 273], standing for

$$\frac{46+\frac{42+\frac{6+\frac{33}{53}}{53}}{53}}{53},$$

as if it meant simply $\frac{3364246}{53535353}$ [ed. Arrighi 1989: 112].[11] The implication is that

[10] This is one of the reasons that this source cannot be Fibonacci’s *liber minoris guise* mentioned in note 2. All conserved treatises of Leonardo, indeed, use the same writing of mixed numbers. In the first instance this only disqualifies the lost work as a source for this particular treatise. However, the argument for the general importance of this “book in a smaller manner” is the similarity of other treatises to the present one on various accounts, e.g. in the presentation of the rule of three – which seems to imply that it breaks down generally.

This argument of course does not invalidate the reasonable assumption that the “book in a smaller manner” treated all or some of the same matters as later *abbaco* books. All we know about it is that Fibonacci says to have borrowed from it an alternative method to treat the alloying of three kinds of bullion for the *Liber abbaci*, and that an anonymous fifteenth-century *abbaco* writer had heard about it and characterized it as a *Libro de merchaanti* (Biblioteca Nazionale di Firenze, Pal. 573, fol. 433^{v}, see [Franci, forthcoming]).

[11] Beyond Arrighi’s familiarity with Fibonacci’s text, two strong arguments speak against the counter-hypothesis that Arrighi did not notice small spaces in the manuscript and committed the error. Firstly, the manuscript *never* explains these composite fractions as does Fibonacci, nor are they ever translated into other fractions, as happens when Fibonacci’s composite fractions of *soldi* are occasionally transformed into *denari* and fractions of these. Secondly, on fol. 108^{v}, Fibonacci’s 100 $\circ\frac{5\,5\,5\,5\,5\,5\,5\,5\,5\,5\,5\,5\,5\,5\,5\,5\,5\,5}{4\,4\,4\,4\,4\,4\,4\,4\,4\,4\,4\,4\,4\,4\,4\,4\,4\,4}$ [ed. Boncompagni 1857: 312], meaning $100\cdot[\frac{5}{4}+(\frac{5}{4})^2+(\frac{5}{4})^3+\ldots+(\frac{5}{4})^{18}]$, reappears in the Umbrian

the compiler never performed these computations and would not have been able to explain them in his teaching. At least those of Fibonacci's problems where such fractions occur are thus taken over as mere external embellishment,[12] no more to be identified with brain gymnastics than looking at it in the TV has to do with genuine gymnastics. But the observation should probably not be retricted to the problems containing composite fractions. For this there are too many borrowed cross-references to matters that are not borrowed, and genuine misunderstandings of sophisticated matters.

Two cases where such misunderstanding is blatant can be found in chapter 21, fols 86^{v}–87^{r}. The first corresponds to a problem which Fibonacci [ed. Boncompagni 1857: 399] solves by means of his letter formalism ("Somebody has 100 *libras*, on which he earned in some place; then he earned proportionally in another place, as he had earned before, and had in total 200 *libras*"). The compiler speaks of two different persons; does not tell that the second goes on with what the first has in total, as he must if the computations shall be meaningful; and eliminates the letters from the text when translating. The outcome is evidently pure nonsense.

The second is a mixed second-degree problem ("Somebody had 100 *libras*, with which he made a travel, and earned I do not know what; and then he received 100 *libras* more from a partnership, and with all this he earned in the

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treatise as $\frac{555555555}{444444444}\ \frac{555555555}{444444444}$ 100 (the fraction being perhaps split over two lines in the manuscript used by the compiler), that is, as a sum of two fractions and an integer. This could not have resulted from an editorial misreading of the manuscript. Instead, it corresponds exactly to an additive juxtaposition of fractions found in chapter 2 (see note 12).

On fol. 136^{r} Fibonacci's $\circ\frac{9\ 9\ 9\ 9\ 9\ 9\ 9\ 9\ 9\ 9\ 9\ 9}{10\ 10\ 10\ 10\ 10\ 10\ 10\ 10\ 10\ 10\ 10\ 10}$ is correctly replaced by $\frac{9}{10}\cdot\frac{9}{10}\cdot\frac{9}{10}\cdot\frac{9}{10}\cdot\frac{9}{10}\cdot\frac{9}{10}\cdot\frac{9}{10}\cdot\frac{9}{10}\cdot\frac{9}{10}\cdot\frac{9}{10}\cdot\frac{9}{10}\cdot\frac{9}{10}$, but this seems to be due to the circumstance that here Fibonacci explains the transformation in detail. Indeed, in the following analogous problem copied from another place in the *Liber abbaci* where no explanation is given, understanding fails completely.

[12] One non-Fibonacci problem contains a composite fraction, but of a wholly different (namely, additive) kind: on fol. 2^{r}, the division of 63 *denari* by 100 is split up, 60 *denari* giving $\frac{3}{7}$, and 3 *denari* giving $\frac{1}{50}\ \frac{1}{100}$. This is obviously well understood.

same proportion as in the first travel, and thus had 299 *libras*"), which Fibonacci [ed. Boncompagni 1857: 399] transforms by means of continued proportions into a rectangle problem which he solves using *Elements* II.6. All letters and lines have disappeared in the translation, as has the Euclidean reference.

A particular difficulty for our compiler is that he does not understand Fibonacci's *regula recta*, the application of first-degree *res*-algebra (apparently not counted as algebra by Fibonacci). Mostly, Fibonacci's alternative solution by means of *regula recta* are simply skipped, but in one place (fol. 83^r) he takes over a *regula-recta* solution from Fibonacci [ed. Boncompagni 1857: 258], promising to teach the solution "per regola chorrecta" (demonstrating thereby that he does not know what *regula recta* stands for); omits the first *res* from Fibonacci's text (the position) while conserving some of the following as *cosa*, obviously without noticing that this *thing* serves as an algebraic representative for the unknown number. Beyond elucidating once again the merely ornamental function of Fibonacci's sophisticated problems in the treatise,[13] this shows that the compiler worked at a moment when even the most elementary level of algebra was still unknown in his environment.

Let us then turn our attention to those chapters which teach matters of real commercial use – that is, to chapters 1–10 and 12–15. As we see, only chapters 10 and 12 contain problems taken over from Fibonacci; moreover, those which are taken over all belong to the most sophisticated and often rather artificial class.

The claim that the treatise is shaped "according to the opinion of master Leonardo Fibonacci" is thus in itself an instance of embellishment.[14] The treatise is certainly no "genuine and proper vernacular" version of Fibonacci's work, "made easier by elimination of the most abstract and theoretical parts", nor is it written in order "to vulgarize, among the merchants, Leonardo's mathematical works". At most, this earliest extant *libro d'abbaco* is one in which, in Giusti's words, "the author limits himself to dig in the mine of examples and

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13 Indeed, the method in question is well explained by Fibonacci in the *Liber abbaci* [ed. Boncompagni 1857: 191] and regularly used after that in chapter 12 [ed. Boncompagni 1857: 198, 203*f*, 207, 213, 258, 260, 264, 280].

14 Unless we take it to refer to the writing of mixed numbers and the use of Arabic numerals, which – given the actual number of borrowed *problems* – does not seem very likely.

problems from the *Liber abaci*, in order to find material he could insert in his own treatise" – but without understanding this material, only as a way to show off in front of others who understood no more. Already in the outgoing thirteenth century, Fibonacci had apparently acquired the status of the culture hero of the *abbaco* culture.

Our compiler certainly *could* have found even the simple material for his basic chapters in the *Liber abbaci* – apart from interest on loans (present only as an element in complex problems) all of it is there. But he may have preferred to use examples referring to the metrologies and exchange rates of his own times and area; alternatively, he may already have had a treatise in hand which was ready for all practical purposes[15] and then have decided to insert into it the embellishments borrowed from the hero[16]; for the last chapter he dug in further sources, some of which are also likely to have surpassed his mathematical wits.[17] We cannot know in exact detail what he did. What we *can* know from the analysis is that the *abbaco* tradition of the outgoing thirteenth century was *no Fibonacci tradition*, even though it was *already a tradition*.

Reverence for glorious fathers

The "genuine and proper vernacular versions" of Fibonacci's works came later, when a few *abbaco* masters felt the ambition to trace the sources of their field (the full or partial translations are listed in [Van Egmond 1980: 363]). A

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15 Whether such a primitive version was written by himself or borrowed wholesale from a precursor we cannot know for sure – but the way concrete mixed numbers are spoken of suggests that he did use borrowed material profusely for the non-Fibonacci parts of his treatise. So does the similarity between his way to introduce the rule of three and the way it its introduced in the *Liber habaci*, cf. below, note 29.

16 An obvious model for this possibility is Bombelli, whose *L'algebra* was already finished in a first version when he discovered Diophantos.

17 I have identified one problem [ed. Arrighi 1990: 119] which with great likelihood come either from the Columbia Algorism [ed. Vogel 1977: 83] or something very close to it, a problem about two kinds of dirty wool that shrink at different rates when washed (not only the story and the numbers ar shared but also offside explanatory remarks). However, *if* the version we find in the Columbia MS was indeed the source, the Umbrian compiler must have understood what went on here (not difficult, indeed), since he adds a remark that this is a subtle method for comparing goods.

couple of translations from the *Liber abbaci*, one of chapters 14–15, another of most of chapter 12 and a little of chapter 13, go back to c. 1350; another translation of chapters 14–15 can be dated c. 1400, as can a translation of the *Liber quadratorum*; a translation of the *Practica geometrie* is dated 1442.[18] This can be contrasted with the total number of extant vernacular mathematical writings made within consecutive 25-year periods according to [Van Egmond 1980: 407–414] (with the proviso that some datings are approximative, and others too early because copied internal evidence may give an early dating to a later text):

1276-0	1301-25	1326-50	1351-75	1376-00	1401-25	1426-50	1451-75	1476-0
1	8	10	6	19	16	39	56	66

The age distribution of surviving complete or partial Latin *Liber-abbaci* manuscripts is not very different from that of the translations; 3 appear to be from the later 13th century, 4 from the 14th, 2 or 3 from the 15th, 3 or 2 from the 16th.[19]

The number of vernacular versions of al-Khwārizmī's algebra (or part of it) turns out to equal the total number of translations from Fibonacci (namely five) – see [Franci & Toti Rigatelli 1985: 28–30] and Van Egmond 1980: 361]. One is from c. 1390, one from c. 1400, and three from the fifteenth century. In several cases, interest in al-Khwārizmī goes together with interest in Fibonacci – obviously, both play the role of (mythical) fathers, those *fontes* which it was not uncommon to look for in the Italian fourteenth and fifteenth centuries. Obviously, all of these together count as almost nothing compared to the total number of *abbaco* manuscripts, and analysis of most treatises from the fourteenth and fifteenth centuries would reveal a picture similar to that of the Umbrian *abbaco* – with the difference, however, that the number of borrowings from Fibonacci would be much smaller, and that the need for showing off beyond one's real mathematical competence was now much less urgent and mainly fulfilled by the display of dubious solutions to algebraic equations of the third and fourth degrees –

18 All translations, we notice, are of sophisticated matters.

19 Menso Folkerts, private communication from 1989. Some of the datings are uncertain or disputed.

a fashion originating somewhere between 1307 and 1328 [Høyrup 2001] within that *abbaco* tradition which was already independent (and independent of Fibonacci) around 1290, and which on the whole remained independent.

Fibonacci and the **abbaco**

But what about Fibonacci himself? He certainly took his inspiration from many sources, some of which can be identified – as we have seen, the notation for ascending continued fractions emulates that of the Maghreb mathematical school, the algebra of the *Liber abbaci* copies creatively but unmistakeably from Gherardo of Cremona's translations of al-Khwārizmī's *Algebra* [Miura 1981], the *Pratica geometrie* from the same translator's version of Abū Bakr's *Liber mensurationum*.[20] Most of his sources, however, are unidentified. If the *abbaco* tradition does not descend from Fibonacci, could then Fibonacci instead have taken an already emerging *abbaco* tradition as his starting point?

The title *Liber abbaci* is irrelevant to the question, since it is definitely not Fibonacci's own title. Already Boncompagni [1854: 88–94] pointed out that he invariably refers to the work as *Liber numerorum* or, in the dedicatory letter of the *Flos*, as his *Liber maior de numero*. The word *abbaco* does appear at least thrice (as a latinized genitive *abaci* / *abbaci*) in the book: in the prologue, where he tells to have pursued *studio abaci* for some days in Bejaïa, as already quoted; when chapter 12 is told to treat of *questionibus abbaci* [ed. Boncompagni 1857: 166]; and when the numerical determination of the approximate square root of 743 is told to be done *secundum abaci materiam* [ed. Boncompagni 1857: 353].[21] At least the latter two occurrences do sound as if something specific is meant, and could well refer to such things as we find in the earliest *abbaco* treatises; but it hardly proves anything.

More informative are certain key phrases that abound in the Umbrian as well as later *abbaco* writings. Very often, problems start by the phrase (I quote

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20 [Ed. Busard 1968]. This treatise is indeed the source for most (if not all) of what Fibonacci is normally taken to have borrowed from Savasorda for his *Pratica geometrie*, as becomes evident as soon as the three texts are compared..

21 The approximate root is found as $27\frac{7}{27}$ by means of a procedure that is familiar from many places, among which both Maghreb sources and *abbaco* treatises.

the Umbrian spellings) *famme quista ragione*, "make this problem for me" or *se ci fosse dicto*, "if it was said to us, ...". Very often, the procedure description ends by a phrase like *e chusì fa' le semeglante ragioni*, "and make similar problems in this way". Often, the procedure description also starts by the declaration that *quista è la sua regola*, "this is its rule".

In the Umbrian *abbaco*, such phrases are particularly copious in problems that are not taken from the *Liber abbaci*, but many are also glued onto *Liber-abbaci* problems without having a counterpart in the original. What is more interesting is that Fibonacci has scattered though rarer instances of the "make similar problems in this way", as if somewhat influenced by the style of an environment where this usage was pervasive. We also find copious references to "the rule of [e.g.] trees", meaning the rule introduced by means of a problem on a tree.

Similar evidence comes from the particular way in which many of the first Umbrian alloying problems but none of its other problems begin (the initial problems of chapter 10, which are not derived from the *Liber abbaci*), namely in the first person singular, "I have silver which contains *n* ounces per pound"; the later problems, those taken from Fibonacci, start in different ways, and so do the alloying problems in the *Liber abbaci* itself – but in one place, in a general explanation [ed. Boncompagni 1857: 143], we find *cum dicimus: habeo monetam ad uncias quantaslibet, ut dicamus ad 2, intelligimus quod in libra ipsius monete habeantur uncie 2 argenti*, "when we say, I have bullion at some ounces, say at 2, we understand that one pound of it contains 2 ounces of silver". It is not credible that the later *abbaco* tradition should have grasped this hint and generalized it (it is also found in other *abbaco* writings and, even more significant, in Pegolotti's *Pratica di mercatura* [ed. Evans 1936: 342–357]); instead, Fibonacci must be quoting – and the only place where such a standard beginning is possible is in *problems* on alloying (the construction "we say, I have" shows that the choice of the grammatical person *I* belongs within the citation).

However, "style" is more than standard phrases and the choice of grammatical person. In a mathematical text it also involves standards of rigour and correctness (etc.). Even in this respect close reading of Fibonacci's text turns out to be revelatory on at least two points.

The first of these concerns his presentation of the method of a single false position ("the rule of trees"). A typical *abbaco* way to make such computations runs as follows:[22]

> The $\frac{1}{3}$ and the $\frac{1}{5}$ of a tree is below the ground, and above 12 *braccia* appear. [...] If you want to know how long the whole tree is, then we should find a number in which $\frac{1}{3}$ $\frac{1}{5}$ is found, which is found in 3 times 5, that is, in 15. Calculate that the whole tree is 15 *braccia* long. And remove $\frac{1}{3}$ and $\frac{1}{5}$ of 15, and 7 remain, and say thus: 7 should be 12, what would 15 be? 12 times 15 make 180, when divided by 7, $25\frac{5}{7}$ results. And as long is the whole tree. And in this way all similar calculations are made.

Even Fibonacci [ed. Boncompagni 1857: 173*f*], as mentioned, uses a tree for this purpose. $\frac{1}{4}$ $\frac{1}{3}$ of it are below the ground, which is said to correspond to 21 palms. He also searches for a number in which the fraction can be found (in this case of course 12). But then he argues that the tree has to be divided in 12 parts, 7 of which must amount to 21 palms, etc. He goes on explaining that there is another method "which we use" (*quo utimur*), namely to posit that the tree be 12. This explanation ends thus:

> therefore it is customary to say, for 12, which I posit, 7 result; what shall I posit so that 21 result? and when this is said, the extreme numbers are to be multiplied, that is, 12 by 21; and the outcome is to be divided by the remaining number.

Already in 1228, perhaps in 1202, it was therefore "customary" to do as the *abbaco* authors were to do in later times. Since nothing is said about this formulation to be customary in some other place, Fibonacci must refer to a custom belonging to a region the reader can be supposed to know about, and to a "we" of which Fibonacci himself is at least a virtual member.[23] Fibonacci feels obliged to present this way, as what "we" are doing, but evidently prefers to avoid falsity in mathematics, and therefore introduces the subdivision into parts.

The second point is a similar reinterpretation of a more direct challenge to mathematical truth. Many Italian *abbaco* treatises, and *all* Ibero-Provençal writings

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[22] I translate from the Columbia Algorism [ed. Vogel 1977: 79]. Cf. above, note 5.

[23] We notice that Fibonacci does not say that this is "what we say in our book in a smaller manner" (the way he refers to it in the place where it is mentioned, and where the reference concerns a particular alloying calculation). The reference to the "costumary" rules out that the formulation is an ellipsis for this fuller phrase.

I have had the opportunity to examine, contain counterfactual rule-of-three problems – either simple, like "if $\frac{2}{3}$ were $\frac{3}{4}$, what would $\frac{4}{5}$ be?" or "7$\frac{1}{2}$ is worth 9$\frac{1}{3}$, what will 5$\frac{3}{4}$ be worth"[24], or more sophisticated, like "if 5 times 5 would make 26, say me how much would 7 times 7 make at this same rate" or "If 9 is the $\frac{1}{2}$ of 16, I ask you what part 12 will be of 25".[25]

The Ibero-Provençal examples are all of the simple type, and all Ibero-Provençal treatises use them to introduce the rule of three: the Castilian *Libro de arismética que es dicho alguarismo* from 1393 (ed. Caunedo del Potro, in [Caunedo del Potro & Córdoba de la Llave 2000: 147*f*]); the "Pamiers Algorism" [Sesiano 1984: 45] from c. 1430; Francesc Santcliment's *Summa de l'art d'Aritmètica*, printed in Barcelona in 1482 [ed. Malet 1998: 165]; and Francés Pellos' *Compendion de l'abaco* from 1492 [ed. Lafont & Tournière 1967: 103-107]. With the Columbia algorism as only exception, all Italian treatises I know which contain counterfactual problems assign less prominent positions to them - either they serve as alternative examples of the rule of three or they stand as isolated number problems.

The *Liber abbaci* [ed. Boncompagni 1857: 170] presents us with two instances, one from each category: "If 7 were the half of 12, what would be the half of 10?", and "If $\frac{1}{3}$ were $\frac{1}{4}$, what would $\frac{1}{5}$ be?". Yet Fibonacci clearly does not like them as they stand, and explains that by the first it can be understood "that the half of 12, which is 6, grows into 7; or 7 is diminished into the half of 12, which is

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[24] The former example is from a fifteenth-century anonymous *Arte giamata aresmetica*, Torino N.III.53 [ed. Rivolo 1983:11f], the latter from the Columbia Algorism [ed. Vogel 1977: 54].

[25] The former example is from Jacopo da Firenze's *Tractatus algorismi* [ed. Høyrup 1999: 39], the latter from Paolo Gherardi's *Libro di ragioni* [ed. Arrighi 1987: 17]. The third Italian treatise written in Provence, the *Trattato di tutta l'arte dell'abacho* (Rome, Biblioteca dell'Accademia Nazionale dei Lincei, Cors. 1875, fol. 32^{v}), contains this: "Let us posit that 3 times 7 would make 23, tell me how much 5 times 9 would make at that same rate". The Columbia Algorism [ed. Vogel 1977: 101, 110–112] contains three examples very similar to Gherardi's, one of them twice. Apart from these I have only noticed a counterfactual *calculation* in ps.-Paolo dell'Abbaco, *Istratti di ragioni*, [ed. Arrighi 1964: 89], in words so close to Jacopo's that it might well descend from him – and then in the *Liber abbaci*, see presently.

6"; about the second he tells the reader that it is as if one said, "$\frac{1}{3}$ of a *rotulo* [a weight unit, c. $2\frac{1}{2}$ *libre*] for $\frac{1}{4}$ of one bezant, how much is $\frac{1}{5}$ of one *rotulo* worth?" If Fibonacci uses a formulation which he feels an immediate need to translate he evidently cannot have invented it himself – but no Arabic treatise seems to contain anything similar, which implies that he has found the formulations in the Romance-speaking area.

All in all we may thus conclude that Fibonacci, though mostly trying to be neutral and to emulate scholarly style, was familiar with a tradition that influenced the style of the later *abbaco* writings heavily.

We do not know with certainty that this environment was located in Italy – as I have argued elsewhere ([Høyrup 2001] and, with a more extensive argument, Høyrup 2003]), Italian *abbaco* algebra, when it emerged, received its inspiration not from Fibonacci but from some non-Italian (probably Ibero-Provençal) environment; the importance of the counterfactual rule-of-three problems in this area and their generally more modest position in the sources from Italy point in the same direction.[26] But the various Italianisms that creep into his text

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[26] Since the early Columbia Algorism is an exception to this general observation, the latter argument is in itself only of limited strength. It is noteworthy, however, that precisely the Columbia Algorism [ed. Vogel 1977: 31*f*] might show us the passage from abstraction to fanciful counterfactuality. A number problem with the structure

$$(n-{}^{1}/_{3}n-{}^{1}/_{4}n)\times(n-{}^{1}/_{3}n-{}^{1}/_{3}n) = n$$

is solved from the false position n = 12, whence 5×5 = 25 should be 12. The text runs "5 times 5 are 25; I want that this 25 should be 12, what would 12 be? Say, if 25 were 12, what would 12 be". By containing counterfactual *calculations* (*and on this account only!* - if we look, e.g., at the treatment of the rule of three, things look quite differently), the *Liber abbaci,* the Columbia Algorism and the three Italian treatises written in Provence in the early fourteenth century form a conspicuous cluster.

Noteworthy is also the following problem from the Columbia algorism [ed. Vogel 1977: 122], "Somebody had *denari* in the purse, and we do not know how many. He lost 1/3 and 1/5, and 10 *denari* remained for him". The same problem (a fairly atypical use of the dress of the purse), only with the unlucky owner of the purse being "I" and the remaining dineros being only 5, is found in the *Libro de arismética que es dicho alguarismo* (ed. Caunedo del Potro, in [Caunedo del Potro & Córdoba de la Llave 2000: 167]). Both solve it by way of the counterfactual question, "If 7 were 10 [respectively 5], what would 15 be?". Since the *Libro de arismética* appears to belong squarely within the Ibero-Provençal group and not to have particular affinities with Italian material, this similarity suggests (e.g., *viadium* / *viagium* for travel, from *viaggio*, *avere* as an occasional translation of Arabic *māl* instead of *census*) as well as the observation that Italian merchants already had an urgent need for such things as are taught in the first 15 chapters of the Umbrian *abbaco* suggests that it encompassed Italy without excluding that it ranged more widely.[27]

We should take note of exactly what Fibonacci tells in the prologue of the *Liber abbaci:* that his father brought him to Bejaïa, where his *studio abbaci* introduced him to the "nine figures of the Indians", that is, to the use of the Hindu-Arabic numerals; nothing is said about methods like the rule of three, partnerships, or alloying.[28] Latin culture, as is well known, had already been introduced to these in the early twelfth century; none the less it is highly likely that whatever commercial teaching went on in Italy during Fibonacci's youth was still based on Roman numerals[29], and that the consistent application of

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that the importance of the counterfactual problems in the Columbia Algorism, far from undermining the importance of the Ibero-Provençal area for the emergence of *abbaco* culture, strenghtens the hypothesis (while leaving it a hypothesis that may possibly be killed off by the appearance of further thirteenth-century material pointing in a different direction).

[27] It may also have encompassed the Arabic world. We know next to nothing about Mediterranean-Arabic mathematics teaching for merchants, but it must have existed. Which was the kind of school in Bejaïa where Fibonacci spent "some days"? Certainly no madrasah, hardly a mosque school. And which was the institutional framework for the teaching and transmission of *muʿāmalāt* mathematics? Probably an institution linked to social groups engaged in *muʿāmalāt*, commercial transactions.

[28] Later, of course, the *regula recta* and the *elchatayn* ("double-false") rule are ascribed to the Arabs [ed. Boncompagni 1857: 191, 318]; but these are higher-level matters that go beyond basic *abbaco* teaching as reflected in chapters 1–15 of the Umbrian treatise and the curriculum of the abbacus schools of Pisa as described by one Cristofano di Gherardo di Dino who flourished in 1428–29 [ed. Arrighi 1967].

[29] A *Liber habaci* (Florence, Magl. XI, 88, fols 1ʳ–40ᵛ, [ed. Arrighi 1987: 109–166], dated by Van Egmond [1980: 115] on the basis of internal evidence to 1310, still gives all integers in Roman numerals – also those in the brief exposition of the place-value system (p. 109) – and all fraction denominations in words. Comparison of its introduction of the rule of three with what we find in the Umbrian *abbaco* shows close affinity between the two. Given the vacillating Umbrian writing of fractions based on Hindu-Arabic numerals within mixed numbers we may perhaps guess its compiler to have worked on the basis of material which was similarly based on Roman numerals and verbal fractions – that is,

Arabic numerals to otherwise familiar matters is what makes his treatise really new (apart of course from its exorbitant scope and its integration of algebra and Euclidean material and of numerous sophisticated variants of many recreational problems); such an interpretation would fit his words better than the belief that *everything* in the book was new to his world. Instead of being the starting point of *abbaco* culture Fibonacci may have been an extraordinary representative who, growing, had grown taller and more conspicuous than any other representative – so tall that Cardano saw nobody but him in the landscape who was worth mentioning, although Cardano's own *Practica Arithmeticae generalis* – which undertakes to set straight what was faulty in *abbaco* tradition – contains much *abbaco* material *not* coming from Fibonacci (see [Gavagna 1999]).

Appendix: detailed description of the Umbrian abbaco

Ch. 1, *de le regole de le tre chose* (fols 1^r–1^v).

Here, the rule of three for integers is introduced together with the tricks to use if one or more of the given numbers contains fractions. Nothing is taken over from Fibonacci, although the *Liber abbaci* contains many problems that *could* have been borrowed.

Ch. 2, *de le chose che se vendono a centonaio* (fols 2^r–3^r).

This chapter gives rules of the type "if something is sold in batches of a hundred pounds, then for each *libra* that the hundred are worth, the pound is worth $2\frac{2}{5}$ d., and the ounce is worth $\frac{1}{5}$ d." (1 *libra* = 20 *soldi* = 240 *denari*).

Nothing is borrowed from Fibonacci, but from the end of the chapter (and, with some exceptions, until the end of the treatise) the writing of mixed numbers suddenly follows Fibonacci's system, the fraction being written to the left of the integral part; until then, the integral part stands to the left.

Ch. 3, *de le regole de pepe che senno* (fols 3^r–4^v).

Problems about pepper and other spices, some of them involving loss of weight due to refining. Nothing is borrowed from Fibonacci.

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that an expression like *d.* $\frac{4}{11}4$ *de denaio* reflects an effort to adapt writings like *gienovino vii et septe ottavj d'uno gienovino* [ed. Arrighi 1987: 125] to Fibonacci's notation.

Ch. 4, *de le regole degle drappe che se vendono a channa e a br.* (fols 4^{v}–6^{r}).

Problems depending on the metrology for cloth. Nothing is borrowed from Fibonacci.

Ch. 5, *de regole de chanbio* (fols 6^{r}–13^{r}).

Mostly on exchange of one coin against another – but also of coin against weighed bullion, silk or fish, and of combination of coins, depending mostly on the rule of three and involving the subdivisions of the *libra*. Nothing is borrowed from Fibonacci. From fol. 7^{r} onward, many results are given in the awkward form "d. $\frac{17}{49}7$ de denaio", "*denari* $\frac{17}{49}7$ of *denaro*", and similarly – obviously arising from infelicitous mixing of Fibonacci's notation with the standard expression "*denari* 7, $\frac{17}{49}$ of *denaro*".[30] The implication is that the compiler has copied this section from another pre-existing written source making use of the standard idiom (which is hardly unexpected). The same construction turns up again in various later chapters, but *never* in problems taken over from Fibonacci.[31]

Ch. 6, *de baracta de monete e denari* (fols 13^{r}–15^{r}).

More complex problems on exchange of coin (and merchandise), involving the (unnamed) rule of five. Nothing is borrowed from Fibonacci.

Ch. 7, *de le regole de marche Tresce* [from Troyes / JH] *e de svariate ragione de lib* (fols 15^{r}–16^{v}).

Similar to chapter 6, but even more complex. Nothing is borrowed from

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[30] This is the notation that is used until that of Fibonacci is adopted on fol. 2^{v} – e.g., "denare 19, $\frac{17}{49}$ de denaio", fol. 1^{r}. In both cases, as we notice, the first time the unit is mentioned it occurs as a plural, the second time as a singular genitive, which excludes a reading of the inverted expression as "*denari* $\frac{17}{49}$, 7 *denari*".

[31] Fol. 7^{r} has an isolated "d. 10, $\frac{6}{7}$ de denaio" betraying the original, and slightly later "d. $\frac{2493}{1200}5$ de denaio". Similar slips are found on fol. 45^{r}, "dr. 1, $\frac{21}{50}$ de denaio", and fol. 134^{r}, "d. 3, $\frac{15}{19}$ de denaio", "d. 8, $\frac{4}{19}$ de denaio". On fol. 57^{r} and again on fol. 121^{r}, whole schemes are organized accordingly. All of these instances are in problems not borrowed from Fibonacci.

Fibonacci.

Ch. 8, *da sapere quante d. de chantra e charrubbe e grana è l'onzia* (fols 16^{v}–17^{v}).
On the subdivision of the ounce, and on the purification of alloyed bullion. Nothing is borrowed from Fibonacci.

Ch. 9, *de conparare bolçone a numero de denare ed a peso de libr* (fols 17^{v}–20^{v}).
Problems on the purchase of alloyed bullion and its evaluation in value of pure metal. Nothing is borrowed from Fibonacci.

Ch. 10, *de regole de consolare ed alegare monete* (fols 20^{v}–29^{v}).
Problems about alloying. After ten simple problems that are independent of Fibonacci follow eighteen, some of them more complex, that are borrowed from the *Liber abbaci* [ed. Boncompagni 1857: 144–158] – in part whole sequences of consecutive problems. At times the copying is so close that Fibonacci's cross-references are borrowed even though they are invalid in the actual context; at times minor variations are introduced, e.g. the conversion of $\frac{101}{163}$ ounce into $7\frac{71}{163}$ [*denari*] (1 ounce is 12 *denari*).

Ch. 11, *de svariate regole che s'apartengono al consolare de le monete* (fols 29^{v}–32^{v}).
Six rather artificial problems of alloying type, five of which are from the *Liber abbaci* [ed. Boncompagni 1857: 159–164].

Ch. 12, *de regole de merto o vero d'usura* (fols 32^{v}–42^{v}).
Problems about loans and interest, first 24 on simple interest, then one (counting a numerical variant, two) problems about composite interest over full years and one on a decrease in geometrical progression; none of these come from Fibonacci (the last problem is structurally analogous to one found in the *Liber abbaci* [ed. Boncompagni 1857: 313], but the solution runs along different lines).[32] In the end comes a section "De sutile regole de prestiare lib. quante tu vuogle ad usura sopre alchuna chosa", about giving a loan in a house which the creditor rents, the excess of the rent over the interest on the loan being discounted from the capital; all 12 problems belonging to this section are borrowed from the *Liber abbaci* [ed. Boncompagni 1857: 267–273].

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[32] In this and in a slightly earlier problem, we also find constructions of the type "d. $\frac{1747}{2561}1$ de denare"

Ch. 13, *de regole che s'apartengono a quille de la usura* (fols 42^v–44^r).

Eight problems somehow involving interest (combined with partnership, discounting, etc.), not derived from Fibonacci. There are a few instances of constructions like "d. $\frac{2493}{1200}$5 de denaio", and also one "dìne $\frac{6}{97}$13 de dìne", "days $\frac{6}{97}$13 of day".

Ch. 14, *de regole de saldare ragione* (fols 44^r–51^r).

Loan contracts containing invocations of God, names and dates, thus real or pretendedly real, leading to the problem of repaying at one moment several loans made within a single year; only simple interest is involved. The whole chapter is independent of the *Liber abbaci*. There are copious instances of expressions of the type "d. $\frac{11}{12}$6 de denaio" in all those problems that permit it, with the implication that this section is copied from a written source, either real contracts or another *abbaco* treatise.[33]

Ch. 15, *de svariate regole de conpagnie* (fols 51^r–58^v).

Various partnership problems, none of which come from Fibonacci. Most of them contain constructions of the type "staia $\frac{5}{8}$90 de staio" (the *staio* is a measure of capacity).

Ch. 16, *de chonpare de chavagle* (fols $58^{v–65}$r).

Ten variations of the "purchase of a horse". The first two are independent of Fibonacci (the second is indeed of "partnership" type, the following eight are taken over from the *Liber abbaci* [ed. Boncompagni 1857: 228–235, 253*f*]. The first two problems contain numerous constructions of the type "d. $\frac{4}{7}$8 de denaio", the others none.

Ch. 17, *de huomene che demandavano d. l'uno a l'altro* (fols 65^r–74^r).

Variations (with changing number of men and conditions) of the problem type "Two men have *denari;* if the first gets *a* of what the second has, he shall

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[33] Since this is the chapter whose dated problems suggest that the treatise was written in c. 1290, this observation reduces the credibility of that dating for the actual treatise; however, the compiler's total ignorance of algebraic terminology (see text just after note 13) supports an early date; so does, to the extent I can judge it, the apparently archaic orthography.

have *p*; if the second gets *b* of what the first has, he shall have *q*", where *a*, *b*, *p* and *q* may be given absolutely or relatively to what the other has. The first six are independent of Fibonacci, then come nine that are taken from *Liber abbaci* [ed. Boncompagni 1857: 189*f*, 198–202], then finally one that is not borrowed from that work.[34]

Ch. 18, *de huomene che trovaro borsce* (fols 74r–79v).

Seven variations on the theme "*N* men find a purse with *denari*; the first says, 'If I get what is in the purse (with / without what I already have) I shall have *p*'; the second says ...", *N* being 2, 3, 4 or 5, and *p* being given relatively to the possession of the other(s). All come from the *Liber abbaci* [ed. Boncompagni 1857: 212–214, 220, 223, 227].

Ch. 19, *de huomene che cholsero denare emsiememente* (fols 79v–82r).

Five problems of the type "*N* men find *denari* which they divide in such a way that ...", *N* being 2, 3, 5 or 6. In several cases "in such a way" regards the products between the shares two by two. All are borrowed from the *Liber abbaci* [ed. Boncompagni 1857: 204–207, 330, 281, this order]. In the end comes a single problem of the type treated in chapter 18, which is independent of Fibonacci.

Ch. 20, *de regole de prochacio overo de viage* (fols 82r–86v).

Fifteen problems "on gain and travelling", about a merchant visiting three or more markets, gaining every time a profit that is defined relatively to what he brought and having expenses that are defined absolutely; the initial capital is found from what he has in the end. All come from the *Liber abbaci* [ed. Boncompagni 1857: 258–262, 266].[35]

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34 Or which at least is not in the 1228 edition as published by Boncompagni; it *could* in principle be one of those problems from the 1202-edition which Fibonacci states [ed. Boncompagni 1857: 1] to have eliminated as superfluous; indeed, no obvious stylistic features distinguish it from the *Liber-abbaci* problems that precede it. It could also come from the *liber minoris guise*.

35 Max Weber *in memoriam*, the most widespread variant of the problem type could be baptized "pre-Protestant merchant's nightmare": at each market, the merchant promises God to give a specific amount to the Church or the poor if God doubles his capital; this happens thrice, after which the pious merchant is bankrupt. The earliest extant appearance of this problem is in Ananias of Širak's seventh-century problem collection [ed., trans. Kokian 1919: 116]; it disappears after the Reformation, in good agreement with the Weber thesis – but it is also avoided by Fibonacci, who may have found it too far removed from

Ch. 21, *de huomene ch'andaro a guadagnare agl merchate* (fols 86^{v}–91^{r}).

Nine problems concerning trade or markets – not all, in spite of the title, about gains (two, indeed, are of the type "a hundred fowls"), but all are from the *Liber abbaci* [ed. Boncompagni 1857: 399, 298, 160, 165*f*, 179, this order]; in some cases it is obvious that the compiler does not understand what he copies, cf. p. 9.

Ch. 22, *de choppe e del suo fondo* (fols 91^{r}–92^{r}).

Three problems about a cup consisting of a cover, a foot, and "el meço" ("the middle"), one part being given absolutely, the others relatively. The problems correspond to a sequence of consecutive problems in the *Liber abbaci* [ed. Boncompagni 1857: 188*f*];[36] the second contains a backward reference to the use of the "rule of the tree" even though this rule, earlier in the *Liber abbaci*, comes later in the present treatise. The same second problem is corrupt, seemingly because the manuscript that is used has employed "$\frac{1}{2}$" as a word sign for *medium* or *meço*, which the present writer repeats but understands as a number.

Ch. 23, *d'arbore o vogle de legne* (fols 92^{r}–93^{r}).

Four problems about a tree, a certain fraction of which is either hidden underground, or added to the tree, the remainder or total being given absolutely. They correspond to a sequence in the *Liber abbaci* [ed. Boncompagni 1857: 174*f*], but the wording of the first problem deviates so much from the *Liber-abbaci* counterpart (and corresponds so well to what is found in other *abbaco* writings, e.g., in the Columbia Algorism) that one may assume the writer to have rewritten this problem from Fibonacci in a familiar style, knowing it also from elsewhere.

Ch. 24, *de vasa* (fols 93^{v}–95^{r}).

Two problems about three respectively four vases, relative relations between whose contents are given (e.g., that the first holds $\frac{1}{18}$ of what the second

what real merchants from his times and town would do.

[36] In order to see that the second problems has a counterpart in the *Liber abbaci* one has to discover (from the subsequent calculation, or from our Umbrian *abbaco*) that the words "ponderet quantum medii" [ed. Boncompagni 1857: 188, line 5 from bottom] should be "ponderet quartum medii".

holds, plus $\frac{1}{3}$ of what the third holds). Both are from the *Liber abbaci* [ed. Boncompagni 1857: 286].

Ch. 25, *de huomene che vonno per via chumunalemente ensieme* (fols 95^{r}–96^{r}).

Two problems about men putting part of their possessions or the total of these in a common fund, redistributing part of the fund arbitrarily and the rest according to given proportions, finding thus their original possessions. Both are from the *Liber abbaci* [ed. Boncompagni 1857: 293, 297], but the anecdote in the first one differs from Fibonacci's version (but probably coincides with the typical tale belonging with the problem).

Ch. 26, *de huomene che portaro margarite a vendere em Gostantinuopole* (fols 96^{r}–97^{v}).

First two problems about carrying pearls to Constantinople and paying the customs, both taken from the *Liber abbaci* [ed. Boncompagni 1857: 203*f*]; next one which combines a dress about precious stones and Constantinople with the mathematical structure of a problem dealing with fishes and commercial duty, neighbouring problems in the *Liber abbaci* [ed. Boncompagni 1857: 276*f*].[37] Finally an independent problem about trade in pearls.

Ch. 27, *de tine e de botte cho' n'esce el vino per gle foramene cho sonno el fondo* (fols 98^{r}–101^{r}).

Six problems on perforated tuns and casks, all from the *Liber abbaci* [ed. Boncompagni 1857: 183–186]. In the first problem, the compiler misrepresents and obviously does not understand the explanation of the procedure given by Fibonacci.

Ch. 28, *d'uno che manda el figlo en Alixandria* (fols 101^{r}–102^{r}).

Four problems on the purchase of pepper, saffron (in the fourth also sugar and cinnamon) for a given total, at given prices and at given weight proportions. All are borrowed from the *Liber abbaci* [ed. Boncompagni 1857: 180].

Ch. 29, *d'uno lavoratore che lavorava enn una uopra* (fols 102^{r}–104^{v}).

First two identical problems about a worker who is paid for the days he works and pays a fine for the days he does not work, solved with different

[37] Given the restricted competence of our compiler one may ask whether these two problems were one in the version of the *Liber abbaci* which he had at his disposal (the 1202 edition?) or in the *liber minoris guise*, given that this book shares material with the *Liber abbaci*.

methods; the two versions are taken from widely scattered places in the *Liber abbaci* [ed. Boncompagni 1857: 323, 160] – the first of them appears to be badly understood by the compiler. Next comes a problem about a complex mode of wage payment, again in two versions with different solutions, both coming from the *Liber abbaci* [ed. Boncompagni 1857: 186, 324].

Ch. 30, *de huomene ch'andano l'uno po' l'altro* (fols 104^v–105^r).

Two problems about two travellers, one going with a constant speed, the other pursuing him with a speed that increases arithmetically. From the *Liber abbaci* [ed. Boncompagni 1857: 168].

Ch. 31, *de regole per molte guise forte e ligiere de molte contintione* (fols 105^r–136^v).

A mixed collection of mainly recreational problems, some from the *Liber abbaci*, others not; several of the latter are simpler versions of problems borrowed from Fibonacci that appear in the preceding chapters. With reference to the pagination in [Boncompagni 1857], the distribution is as follows: I indicate by "≠" that superficial similarity seems to suggest a borrowing but closer inspection shows instead that both writers draw on a common fund of basic problems and variations:

p. 273, p. 273, p. 274, p. 297, p. 298, p. 298, p. 283,[38] p. 329, p. 312, p. 182, p. 182, indep., indep., indep. (≠p. 307), indep. (≠p. 323, ≠p. 160),[39] indep., indep.,[40] indep. (≠p. 179), indep., indep., indep., indep.[41], p. 403, indep., indep., indep., indep., indep., indep., indep. (≠176*f*), p. 181, indep. (≠172),[42] indep. (≠174), indep.,[43] indep., indep.,[44] indep.,[45] indep.,[46] p. 177, p. 182,[47] p. 274, p.

38 The "rabbit problem", transformed into a "pigeon problem" with no other change, and with a reference to a marginal diagram that is indeed found in the *Liber abbaci* but not in the present treatise.

39 Apparently based on a source which is understood and copied badly.

40 Meaningless as it stands, probably resulting from defective copying of a source. To be solved "sença regola [...] a palpagione e per apositione falsa".

41 This is the problem that is borrowed from the Columbia Algorism or its closest kin, cf. note 17.

42 Apparently based on a source that is copied thoughtlessly.

43 Solved wrongly.

44 A question touching at a real-life problem for long-distance trade which is rarely mentioned in *abbaco* treatises: a ship beating up against the wind.

45 An eternal calendar.

311, p. 316,[48], p. 313, p. 309,[49] p. 311, indep., indep., indep., indep., indep., indep., indep., indep., indep., indep., p. 132,[50] p. 133, indep.,[51] (≠167),[52] (≠167),[53] (≠166),[54] indep., indep., indep., indep. (≠p. 179), indep., indep., p. 283,

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46 This problem type is often found in *al-jabr* treatises: to divide a given number (mostly 10, here $16\frac{1}{4}$) into two parts with a given ratio.

47 About ships that encounter each other. The compiler has added names to the points of departure and destination (Genoa and Pisa).

48 Contains a cross-reference to the problem that precedes – in the *Liber abbaci*. Here it follows.

49 The chess-board problem. The beginning copies Fibonacci in a way that suggest failing understanding.

50 Makes use of the "rule of five" but without explaining what goes on (Fibonacci explains).

51 Another instance of sloppy copying from a source – the problems starts, in word-for-word translation, "There is a well and a serpent deep 90 palms, by day $\frac{3}{3}$ palms and ascends and by night descends the fourth". Apart from the displaced and superfluous words, "$\frac{3}{3}$" should be "$\frac{2}{3}$".

52 The rule for the summation of square numbers from 1^2 to 10^2, found as $10\cdot(10+1)\cdot(10+[10+1])/6$. The same computation is found in the *Liber abbaci* [ed. Boncompagni 1857: 167], but the formulations are too different to make a borrowing plausible. The general case (with the corresponding formula) is proved in Fibonacci's *Liber quadratorum* [ed. Boncompagni 1862: 262], but nothing in the formulations suggest the compiler to have used that work.

53 The rule for the summation of odd square numbers from 1^2 to 11^2, found as $11\cdot(11+2)\cdot(11+[11+2])/(2\cdot6)$. The *Liber abbaci* [ed. Boncompagni 1857: 167] finds the sum $1^2+3^2+...+9^2$ according to the same formula, but explaining that the factor 9+2 is the following member of the sequence of odd numbers, and that the divisor 2 is the distance between the squared numbers. Once again, a general proof is found in the *Liber quadratorum* [ed. Boncompagni 1862: 263], but nothing in the formulations suggests the compiler to have known that work.

54 The rule for the computation of 1+2+...+99, found as the product of the last member by its half rounded upwards! The *Liber abbaci* [ed. Boncompagni 1857: 166] gives two general formulae, either half the number of terms multiplied by the sum of the extremes, or half this sum multiplied by the number of terms, and one less general for sums of the type $p+2p+...+np$; all Fibonacci's numerical examples differ from the present one. The *Liber quadratorum* [ed. Boncompagni 1862: 265] contains the rule that the sum of a number n and other numbers pairwise equidistant from it (i.e., $n+(n+d_1)+(n-d_1)+...+(n+d_p)+(n-d_p)$)

indep., indep., indep.

A few of the independent problems contain expressions like “d. $\frac{2}{7}$4 de denaio”, (fol 110^{v}), “d. $\frac{1}{3}$9 de denaio” (fol. 112^{r}); “dì $\frac{11}{30}$354 de d.” (fol. 122v).

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equals the product of the number (n) and the number of terms ($2p+1$). It is hardly necessary to argue that this generalization of the summation of an arithmetical series with an odd number of terms was not used by our compiler.

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