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The founding of Italian vernacular algebra

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CONTENTS

Jacopo or ps.-Jacopo?

In [1929], Louis Karpinski published an article on "The Italian Arithmetic and Algebra of Master Jacob of Florence, 1307". Here, he described the Vatican manuscript Lat. 4826 (henceforth **V**), while pointing out that another manuscript of Jacopo's work, Riccardiana Ms. No 2236 (henceforth F), had been mentioned by Boncompagni in 1883, and discussed by the librarian of the Riccardiana in 1754. He had not seen this version of the treatise, and his sources did not allow him to discover the differences between the two manuscripts – in particular the absence of the algebra section from **F**.

What he did discover were some of the differences between the algebra of **V** and earlier Latin writings on the subject – the translations of al-Khwārizmī and Abū Kāmil as well as the *Liber abbaci*. He pointed out that both the examples and the order of the cases deviate from the earlier models; that one of the "new" examples is already found with Mahāvīra though with other numbers;[1] he also mentioned the presence of rules for solving 14 reducible third- and fourth-degree equations, but did not tell explicitly that **V** gives no geometrical proofs for its solutions of the mixed second-degree problems.

1929 fell in the middle of a period where the interest in European medieval mathematics was at its lowest ebb since 1840 (perhaps since the very Middle Ages); moreover, few pieces of *abbaco* mathematics were known by historians of mathematics,[2] – so few indeed that they were

[1] He did not mention, however, that Mahāvīra's method [trans. Raṅgācārya 1912: 151] is quite different (and much more straightforward), which diminishes the significance of the observation.

[2] [Libri 1838: III, 302–356] contains extracts from Piero della Francesca's *Trattato d'abaco* and from the *Aliabraa-Argibra*; Karpinski [1910] had described a third treatise, and Elisabeth Buchanan Cowley a fourth in a publication from 1923 which I have not seen, but which is mentioned in the preface to the edition of the text in [Vogel 1977]. I may have overlooked a few other items, but probably not much with which historians of mathematics were familiar at the epoch.

not recognized to constitute a mathematical genre of their own.[3] There may therefore be little reason to wonder that Karpinski's observations went unnoticed.

In [1976: 488*f*, 517], Warren Van Egmond described both manuscripts, referring also to Karpinski's description of **V**. In [1980: 166*f*], he described a third manuscript, Trivulziana No. 90 (Milan; henceforth **M**; written in Genova around 1410), whose contents is similar to F.

None the less, [Van Egmond 1978] identifies Paolo Gherardi's *Libro di ragioni* from 1328 as "the earliest vernacular treatment of algebra" without referring to Jacopo's treatise; indeed, as Van Egmond tells me in a personal communication, the fact that **V** was written in the mid-fifteenth century and contains rules for the fourth degree not present in Gherardi's treatise makes him conclude "that the algebra section of Vat.Lat. 4826 is a late 14th-century algebra text that has been inserted into a copy of Jacopo's early 14th-century algorism by a mid-15th-century copyist".

As we shall see, the differences between the algebra of **V** and the Latin predecessors (which, on their part, are close to al-Khwārizmī and Abū Kāmil) are even greater than pointed out by Karpinski. It is therefore of some importance for our understanding of the development of European (and, as it turns out, Arabic) algebra to decide whether it goes back to Jacopo da Firenze (or at least some close contemporary of his) or to some late ps.-Jacopo.

The Vatican manuscript

Van Egmond [1980: 224] describes **V** in these terms:

> s. XV (c. 1450, *w*[atermark]), Holograph *libreria* treatise
> Paper, 4° gr[ande], 286×203 mm., 72 cc num. orig. 1–59, 59–66, 69–73, plus 2 enclosing guard sheets. [...]. Single hand, a neat semi-cursive Gothic bookhand in 1 col. press-ruled 194–205×118 mm., 32 regularly- spaced lines. Dark brown ink with alternating red and blue initials, some decorated; capitals shaded in yellow on 1r, 33v–41r, lined in red on 41v–42r; some titles in red. Tables on 2v, 3r, 4v–14v outlined in red; many colored diagrams and drawings in the

[3] Cf. Eneström's attack [1906] on Moritz Cantor, who was at least on the track, and the general acceptance of Eneström's argument.

margins.

The incipit reads as follows:

> Incipit tractatus algorismi, huius autem artis novem sunt speties, silicet, numeratio, addictio, subtractio, ⟨mediatio,⟩[4] duplatio, multiplicatio, divixio, progrexio, et radicum extractio. Conpilatus a magistro Iacobo de Florentia apud Montem Phesulanum, anno domini M°CCC° VII° in mensis septenbris.

The rest of the manuscript is in Tuscan with a somewhat Latinizing orthography (*dicto, facto, septimo, scripto, exemplo*, etc. – none of them used quite systematically)[5]. A number of non-standard spellings also seem to reflect the Provençal linguistic environment of Montpellier, the place where the incipit tells the treatise was made, or some other northern region: *el* or *lo* almost consistently instead of *il*,[6] consistently *sera* or *serra* instead of *sara* (i.e. *sarà*) and almost consistenly *mesura* instead of *misura*; mostly *de* instead of *di*, and occasionally *que* instead of *che* (both as an interrogative and as a relative pronoun); mostly *remanere* (with declinated forms) instead of *rimanere*, and mostly *vene* instead of *viene*; mostly also *doi* or *doy* instead of *due*, occasionally *dui* or *duy*; mostly *amendori*, or occasionally *amendoi/-dui*, instead of *amendue*; ten, when not written with numerals, is almost always *dece*; *-ximo* is used instead of *-simo* as ordinal suffix (and *-x-* also elsewhere for *-ss-*); *lira/lire* are *libra/libre*.[7] The copyist seems to have

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[4] Inserted in agreement with **F**, and needed in order to fill out the number of nine species. In all quotations, ⟨ ⟩ is used to indicate scribal omissions, and { } to mark superfluous passages (e.g., dittographies). Editorial comments are in [ ].

[5] We also encounter Latinizing hypercompensations: thus for instance *librectine* (fol. 4^v), *cictadino* (fol. 36^r), *soctilita* (fol 1^r), *tucto, rocto* (both regularly).

[6] *Lo* (occasionally *lu*) appears in all cases where modern Italian requires it, and is used inconsistently before *s+vowel, d, m, p, q, r,* and *t*.

[7] **F** [ed. Simi 1995] has an occasional *el* but mostly *il*, and as far as I have noticed no *de*. I have observed some instances of *ke* but no *que* (the occurrences of *que* and *ke* in the two manuscripts are independent of each other, when parallel passages are compared). In parallel passages, *remanere* (etc.), *vene* and *mesura* in **V** correspond to *rimanere* (etc.), *viene* and *misura* in **F**, *doi/doy* in **V** to *due* in **F**, *amendore* in **V** to *ambendue* in **F**. **F** has *diece* for ten, and speaks of *lire* (and *danaio* for *denaro*; but singular *libra* or *livra*), and has none of the Latinizing spellings of **V**.

Also likely to point to a Provençal (or French or Iberian) environment is the aimed at fidelity toward his model in this and other respects: at times he corrects a spelling, even though both the new and the old spelling are present elsewhere in the text, which suggests an aspiration for orthographic accuracy (but also shows that he did commit errors on this account, some of which he will probably not have noticed); on fol. 39ʳ, where he leaves a sequence of open spaces of c. 2 cm instead of numbers, he makes a note in the margin, "così stava nel'originale spatii". The insertion of forgotten words above the line makes it plausible that the finished copy was collated with the original.[8] On fols 46ᵛ–47ʳ there is evidence that not only the ultimate but also the penultimate copyist tried to be faithful: fol. 46ᵛ starts by telling that a section on silver coins has been omitted by error and is inserted "de rimpecto nel sequento foglio" (it follows indeed on fol. 47ʳ) – but the organization of the page shows that this passage was not inserted after the writing of the following section on "le leghe de monete picciole", and thus that it was present (together with a mark ⫳ indicating the location of the omitted section) in the original used by the ultimate copyist, who followed this original rather than running the risk of accumulating more errors in an attempt to repair the mistake.

Evidently, many abbreviations are used in the manuscript, including abbreviations for *libre*, *soldi*, *denari*, and *braccia*. What is remarkable in comparison with later *abbaco* manuscripts is that key terms for mathematical operations are *never* abbreviated – neither *più* or *meno*, nor *radice*, *cosa*, *censo* or *chubo*.[9] The absence even of as simple an abbreviation as *cēso* for *censo*

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use of *ha* (written *a*) or *si ha* instead of *c'è* in V. This usage is absent from F.

The Florence manuscript of Paolo Gherardi's algebra [ed. Van Egmond 1978], also originally written in Montpellier, replaces *il* by *el* or *lo* as does V, but has *rimanere* and *di*. Two and ten are written with numerals and are hence uninformative. It has some Latinizing spellings, genuine as well as hypercompensations, which may thus be taken to characterize the environment rather than individual propensities of the writers.

[8] It will come as no surprise that the elaborate initials were also inserted after the completion of the text – but it also follows from the omission of an initial on fol. 17ʳ and of another on fol. 42ʳ. The insertion of a missing passage in the margin of fol. 48ʳ may be due to the same hand as the initials.

[9] In contrast, in the tabulation of degrees of fineness of gold coins on fol. 46ʳ, "meno" is abbreviated ⓜ: "charati 24 ⓜ $^1/_5$ per oncia", etc. This abbreviation appears to

can only be a consequence of a deliberate choice.

Remarkable is also a less standardized technical vocabulary than in other treatises – *el diametro*, for instance, may also be both *el dericto de mezzo* and *el mino lungho*. That a number m falls short of another number n by an amount p may not only be expressed "da 7 infino in 3 menoma 4" (**21**.8, fol. 50^{r}) – the standard expression of **F** – but also (in parallel passages) "da 7 infino in 4 mancha 3" (*ibid.*) or "da $9^{1}/_{2}$ infino in $8^{1}/_{2}$ à uno" (**21**.6, fol. 49^{r}), with the variant "da 4 a 7 si à 3" (**21**.4, fol. 48^{r}).

The Latin incipit, the Latinizing spellings and a reference to Boethius's *Arithmetic* in the introduction should probably not be taken as indications that Jacopo was a university scholar, only that he moved in an environment where scholars and mathematical practitioners were in contact (other indications suggest that this was indeed the situation in Montpellier – cf. [Hahn 1982: xxi*ff*); later on he demonstrates repeatedly not to know the difference between *proportio* and *propositio*. The religious invocations of the introduction correspond well to the style of other *abbaco* writers (and of Arabic writings) but only enter the style of more scholarly work in the form of gentle parody.[10]

The structure of **V** is as follows:

1^{r}–1^{v}	**1.**	Incipit and general introduction.
2^{r}	**2.**	Introduction of the numerals and the role of *zero*.
2^{v}–3^{r}	**3.**	Tabulated writing of the numbers 1–49, 50, 60, ..., 100, 200, ..., 1000, 6000[11], ..., 10000, 20000, ..., 100000, 200000, ...1000000, with corresponding semi-Roman[12] writings.
3^{r}–4^{v}	**4.**	Explanation and exemplification of the place-value principle.
4^{v}	**5.**	Introduction to the multiplication tables.
4^{v}–9^{r}	**6.**	Multiplication tables, including multiples of *soldi* expressed

have belonged specifically to the domain in question, cf. [Vogel 1977: 11].

[10] I know of two examples: *Liber Jordani de triangulis*, a student *reportatio* of a series of lectures probably held by Jordanus himself [Høyrup 1988: 347–351], and Chuquet's trinitarian argument for the title of his *Triparty* [ed. Marre 1880: 593].

[11] The row containing the numbers 2000–5000 is evidently omitted by error.

[12] E.g., 600 is ^{c} .

		in *libre* and *soldi*.
9^r–12^v	**7.**	Tables of squares; 1×1, ..., 100×100, 110×110, ..., 990×990, 1000×1000, and $1^1/_2\times1^1/_2$, ..., $19^1/_2\times19^1/_2$.
12^v–14^r	**8.**	Examples of divisions (9 successive divisions of 16-digit numbers by 2, then by 3, 4, ..., 19, 23, 29, 31, 37, 41, 43, 47, 48 (according to the heading the numbers that are *più necessarii*).
14^v	**9.**	Graphic schemes that serve the multiplication, addition, comparison,[13] division and subtraction of fractions (in this order).
15^r–17^r	**10.**	Examples explaining the addition, subtraction, multiplication and comparison of fractions.
17^r–18^v	**11.**	The rule of three, with examples.
18^v–19^v	**12.**	Computations of non-compound interest.
19^v–21^v	**13.**	Rule-of-three problems involving metrological conversions.
21^v–30^r	**14.**	Mixed problems, including partnership and genuine "recreational" problems.
30^v–36^r	**15.**	Practical geometry: Rules and problems involving the diameter, perimeter and area of a circle; the area of a rectangle (and, erroneously, of the regular pentagon); and the "rule of Pythagoras". With approximate computation of square roots.
36^v–42^r	**16.**	Rules and examples for algebra until the second degree.
42^r–43^r	**17.**	Rules without examples for reducible third- and fourth-degree equations.
43^r–43^v	**18.**	A grain problem of alligation type.
43^v–45^v	**19.**	Second- and third-degree problems on continued proportions (dressed as wage problems) solved without the use of *cosa-censo* algebra.
45^v–47^r	**20.**	Tabulated degrees of fineness of coins.
47^v–50^v	**21.**	Alligation problems.
50^v–$58_{bis}{}^v$	**22.**	Further mixed problems, including practical geometry. In part cross-referenced variations or transformations of problems from chapters **14–15**, in part new types.[14]

[13] Seeing which of two is greater, and finding the remainder; kept apart from subtraction though evidently solved in the same way.

[14] In her edition, Annalisa Simi divides **F** as follows:

Incipit, Prologo	corresponding to	**V.1–2.**
Tabella 1–2,	corresponding to	**V.3–4.**
Chapter I,	corresponding to	**V.6–7.**
Chapter I,	corresponding to	**V.6–7.**

In the following, passages in **V** will be referred to as **V.A.*n*** (or, if there is no doubt that **V** is spoken about, simply **A.*n***), where **A** is the chapter number in Arabic numerals and *n* the section number within the chapter (counted in agreement with the initials). Similarly, references to **F** will have the form **F**.R.*s* (or simply R.*s*), where R is the chapter number in Roman numerals and *s* the section number within the chapter (both counted as in [Simi 1995]). Neither manuscript contains any of these numbers.

The text is characterized by interspersed personal and colloquial-pedagogical remarks – for instance:

(a) Abiamo dicto de rotti abastanza, però che dele simili ragioni de rotti tucte se fanno a uno modo e per una regola. E però non diremo più al punte. (**11.1**, fol. 17^r).

(b) Et se non te paresse tanto chiara questa ragione, si te dico che ogni volta che te fosse data simile ragione, sappi primamente ... . (**14.19**, fol. 26^r).

(c) Una torre ... sicomo tu vedi designata de rinpetto. (**15.9**, fol. 31^v; similarly *passim*).

(d) Ora non si vole agiungere inseme come tu facesti ⟨in⟩ quella da sopra. (**15.10**, fol. 32^r).

(e) Fa così, io t'ò anche dicto de sopra, ogni tondo, a volere sapere quanto è el suo diametro, si vole partire per 3 e $^1/_7$. (**15.11**, fol. 32^r).

(f) Et abi a mente questa regola. In bona verità vorrebbe una grande despositione; ma non mi distendo troppo però che me pare stendere et scrivere in vile cosa; ma questo baste qui et in più dire sopra ciò non mi vo' stendere. (**16.14**, fol. 40^v).

(g) Fa così, et questa se fa propriamente come quella che tu ài nanzi a quella ragione de sopra a questa, et in questa forma. Et però non me stendarò in si longho dire como feci in quella. (**21.8**, fol. 49^r).

(h) Et però ho facta questa al lato a quella,[15] che tu intende bene l'una et l'altra,

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Chapter II,	corresponding to	**V.8–9**.
Chapter III,	corresponding to	**V.10**.
Chapter IV,	corresponding to	**V.11–13**.
Chapter V,	corresponding to	**V.14**.
Chapter VI,	corresponding to	**V.15**.
Chapter VII,	corresponding to	**V.20–21**.

F contains no counterparts of **V.5**, **V.16–19** and **V.22**.

[15] In the preceding problem, the area of the circle is found as $1\text{–}^1/_7\text{–}^1/_2{}^{\cdot 1}/_7$ times the square on the diameter; in the actual problem, it is found as $^1/_4$ of the product of diameter and circumference.

et che l'una et l'altra è bona reghola. Et stanno bene. Et così se fanno le simili ragioni. (**22**.6, fol. 52^r).

A mathematical particularity of the manuscript is the use of the partnership as a functionally abstract representation of proportional sharing. In **14**.9–10 (fols 23^v–24^r) it is used to determine the shares of a heritage; in the algebra section, a sub-problem of the partnership problem **16**.3 (fol. 37^r) is represented by means of *a different* partnership; **21**.4 (fol. 48^r) introduces it as a basic tool for alligation computations, which is referred to in **21**.6 (fol. 49^r) and used again explicitly in **21**.8 (fol. 50^v); in **22**.1 (fols 50^v–51^r), a partnership problem where the participants in the *compagnia* do not enter at the same time, imagines a different partnership where the shares are the respective interests which the investments would have earned (and thus proportional to the product of investment and time). The latter type (a modest generalization) is not uncommon – I have noticed it in the fifteenth-century Provençal treatise from Pamiers [ed. Sesiano 1984: 47], with Pier Maria Calandri [ed. Arrighi 1974: 36*f*], and in a fourteenth-century *abbaco* from Lucca [ed. Arrighi 1973: 75, 77] (in the latter also with recourse to the virtual interests, whereas the others just multiply time and investment); and with Paolo Gherardi [ed. Arrighi 1987: 38]; but the general use of the structure as an abstract model I have observed nowhere else.

V also exhibits a rhetorical particularity. When explaining the reason for a particular step, it regularly ascribes to the "tu" of the text such knowledge or conditions that were originally stated by its "io" ("perché tu di' che ...", etc). Examples of this is found in all chapters which offer the occasion, that is, **14–19**, **21** and **22**.

The manuscript is beautifully illustrated. Some of the illustrations are neutral and to the point (circles with indications of the measures of diameter and perimeter, etc.), but many are not (most aberrant is a beautiful plant with flowers, stalk and root along with the rule for approximating square roots).

The Florence versus the Vatican manuscript

I have had no opportunity to examine **M**, but I have been able to confront **V** with Annalisa Simi's edition of **F** [1995].[16]

The most obvious difference is the absence of sections **16–19** and **22**; they are also lacking in **M**, cf. [Van Egmond 1980: 166*f*]. Two obvious possible explanations are at hand. Either a copyist has inserted extra material into **V** (or a precursor in the stemma), as supposed by Van Egmond; or another copyist has deleted some sections when producing a common precursor for **F** and **M** (since exactly the same sections are lacking in both of these, independent abbreviation is unlikely).

But there are differences at numerous other levels which, together with partial and complete agreements, allow us to decide the question and to conclude that **V** is fairly faithful to the original, and **F** a free adaptation.

One clue is the way the illustrations are referred to. **V** habitually refers to the illustrations and diagrams in the margin in words similar to those of quotation (c) (above, p. 7). **F** has most of the same illustrations (somewhat fewer, and none which are not in **V**), but does not always give a reference in the text; when it does, the reference is located after the solution of the problem, whereas **V** gives it after the statement. This excludes simple expansion or abbreviation and again leaves us with two possibilities: either *V* is a rewriting aiming at greater completeness and consistency, or **F** is a rewriting aiming at conciseness; so far, both possibilities remain open.

But some details imply that the manner and the illustrations of **V** are those of the original. Firstly, we may notice the following passage in **F**.VII.7 (p. 37): "simigliantemente il mostriamo materialmente per figure come se fae lo detto allegamento". In the corresponding location in **V.21**.7 (fol. 49^{v}), we read "Et simile(mente) porremo la figura. Nel modo si fa como abbiamo facto de sopra nell'altra ragione". In **V**, the promised diagram is present – but it is absent from **F**. Secondly, both **V.15**.21 (fol. 34^{v}) and the counterpart **F**.VI.17 (p. 30) contain a drawing of a tent, and both give a reference; but whereas the illustration of **V** has the conic form presupposed by the problem, that of **F** is so glaringly discordant with the description in the text that Annalisa Simi inserts a "(sic!)"

A number of other features also lead to the conclusion that **V** is close to an original of which **F** is an adaptation.

[16] When quoting this and other published editions, I follow the orthography of the edition.

Most striking is the shift from "io" to "tu" when reasons for a step are given. As told, it characterizes all chapters of **V** which allow it (**14–19**, **21**, **22**). Of these, **16–19** and **22** have no counterpart in **F**. In the counterparts of **14** and **15** (**F**.V–VI), no similar shift is found – but in the counterpart of **21** (**F**.VII), it turns up regularly in places where it is also found in **V**, but not in those where **V** does not make use of it (except an occurrence of VII.2, the counterpart of which has been omitted from **V**).[17]

At times the two manuscripts differ in the way a problem is dressed. On at least two occasions, however, **F** betrays descent from the formulation of **V**:

(i) **V.14**.30 (fol. 29r) starts "Egli è uno muro, el quale è lungho 12 braccia e alto sette. Et grosso uno et $^1/_4$". The counterpart **F**.V.34 (p. 25) has instead "egli è un terreno lo qual è ampio 12 braccia, cioè uno muro, et è alto braccia 7 ed è grosso braccia 1 et $^1/_4$". Obviously, the writer starts by changing the original wall into a terrain without having read the whole problem, and then discovers that the presence of a height makes the change impossible, and corrects himself.

(ii) **V.14**.3 (fol. 22r) starts "Uno à a'ffare uno pagamento in Bolognia". In **F**.V.3 (p. 17) we find instead "I'ò a'ffare uno pagamento im Bologna" – but afterwards **F** follows **V** in the choice of grammatical person.[18]

The treatment of approximate square roots ($\sqrt{a} \approx n+^d/_{2n}$, $a = n^2+d$) is informative on several levels. Least decisive but still in accordance with the way illustrations are referred to is the contrast between the rather systematic habit of **V** of pointing out that this is *not* precise ("non è appunto") and the much more scattered observations of the same kind in **F**. Moreover, after having presented the rule, **V.15**.13 tells that its outcome "serà la più pressa radice", while **F**.VI.10 (p. 29) believes it to be "radice

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[17] Similar shifts seem to be very rare elsewhere. I have indeed only noticed one instance, namely in the late fourteenth or early fifteenth *Libro di conti e mercatanzie* [ed. Gregori & Grugnetti 1998: 50]. In most cases, however, this treatise (whose pedagogical pretensions are comparable to those of **V**) refers such explanatory information to "noi" and not to "tu" (a possibility which is also used regularly in **V**).

[18] The fact that this choice is not very consistent may have called forth the unsuccessful emendation.

vera o piue pressa", as if the occasional reference to its only approximate character is mere lip service not supported by full understanding.

But there is more to square roots. **V** not only mentions repeatedly that $n+{}^{d}/_{2n}$ is merely "the closest approximation" but also shows time and again (in actual numbers) that $(n+{}^{d}/_{2n})^2$ exceeds a by $d^2/(4n^2)$. In **15.22** (fol. 35^r) it therefore gives the "improved" value $\sqrt{108} = 10^2/_5-{}^4/_{25}$, in **15.25** (fol. 36^r) correspondingly $\sqrt{569} = 23^{20}/_{23}-{}^{400}/_{529}$ – and in **15.20** (fol. 34^v) $\sqrt{33^1/_3} = 5^7/_9-{}^4/_{18}$.[19]

F contains no counterpart of **V.15.22**, and the counterpart of **V.15.25** gives $\sqrt{569}$ as $23^{20}/_{23}$ without pointing out that the value is approximate (and in the following section it gives another example of the rule, still as if it were exact). Once more, this might mean that the fallacy of **V** has been added onto a sounder stem, or that it has been eliminated in **F**. **F**.VI.18 (p. 31), however, the counterpart of **V.15.20**, tells $\sqrt{33^1/_3}$ to be "5 et ${}^5/_6$ meno ${}^{17}/_{54}$ non appunto". $5^5/_6$ is obviously found in the usual way, as $5+(33^1/_3-25)\div(2\cdot5)$. If the fallacious procedure of **V** had been used to find the correction, it would have been $(5^2)/(6^2)$. The actual correction instead is $(2\cdot5^5/_6)/(6^2)$, which appears to mix the formula of **V** with the meaningful correction $({}^5/_6)^2\div(2\cdot5^5/_6)$.[20] The only explanation seems to be that the writer of **F** (or a precursor) will have seen that something is wrong in the correction of **V**, and that he has tried at one point to repair by having recourse to another formula – but unfortunately remembering or applying it wrongly.

These observations should already suffice to show that **F** is a remake, and **V** much closer to what Jacopo had written in Montpellier in 1307 (as also suggested by the orthography of the two manuscripts, with all the provisos which are needed when medieval orthography is used as an argument); for stylistic convenience I shall henceforth take this result for

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19 Obviously determined in this way: $\sqrt{33^1/_3}$ is found as ${}^{\sqrt{300}}/_3$, where $\sqrt{300}$ is approximated from $n = 18$, which gives $5^7/_9$ as the first approximation. Since $(5^7/_9)^2 = 33^1/_3+{}^4/_{81}$, the fallacious rule of **15.22** and **15.25** would hence give the value $5^7/_9-{}^4/_{81}$, which is miswritten $5^7/_9-{}^4/_{18}$ (a similar inversion of digits is found in **14.3**, fol. 22^r).

20 This formula is described by al-Qalaṣādī [ed., trans. Souissi 1988: 61] – and, in ambiguous terms, by ibn al-Bannāʾ [ed., trans. Souissi 1969: 79].

granted. But they do not exhaust the list of characteristic differences between the two manuscripts that point in the same direction.

The use of the partnership as a functionally abstract representation of proportional sharing was mentioned on p. 8. The same idea turns up in F.V.10, the counterpart of **V**.14.9–10, but nowhere else in **F** – in particular not in the alligation chapter, where **V** uses it consistently.

In **V**, areas (and volumes!) are mostly expressed explicitly in *braccia quadre*, not simply in *braccia*. In **F**, this usage is less frequent – and in **F**.VI.16 the writer can be seen to be so little attached to it that he misconstructs the phrase of **V**.15.19 (fol. 34r), "vo' sapere quante braccia quadre è tucto questo terreno", as "dimmi quanto è tutto quel terreno quadro", even though the terrain in question is *not* a square but a regular pentagon.[21]

V, as pointed out, is stuffed with personal and colloquial-pedagogical remarks. **F** has much fewer of these, and their style is less colloquial. Of the quotations (a) to (h), only (c) has a genuine counterpart (**F**.VI.8, p. 28): "io ti mostro qui appresso la forma per meglio intendere". (e) is reduced to "Fa cosie. Et quest'è la sua propria regola. Parti 100 per 3 1/7 in questo modo" (**F**.VI.10, p. 28). (f) and (h), of course, but only these, belong in chapters without any counterpart in **F**. Similarly, the general introduction to the multiplication tables (**V**.5), the specific introduction to the "librettine magiori" (**V**.6.2) and the introduction to the divisions (**V**.8.1) are absent from **F**. Absent from **F** are finally a number of metatheoretical explanations – for instance the explanation which **V**.15.2 gives after telling how to find a circular diameter from the perimeter, and vice versa: "Et se volissi sapere per che cagione parti et moltiplichi per 3 e $^1/_7$, si te dico che la

[21] Both manuscripts find the area as $3s^2-s$, s being the side. This nonsensical formula is certainly derived from the formula for the n'th pentagonal number, $^1/_2\cdot(3n^2-n)$ by omission of the halving.

I have noticed Jacopo's formula in Tommaso della Gazzaia's *Pratica di geometria e tutte misure di terre* [ed. Nanni 1982: 24*f*] from c. 1400 (where, however, somebody has added the note "non e vera"). Tommaso's first example is based on the side 8, as both **V** and **F**. Other problems in Tommaso's treatise also repeat Jacopo with the same numerical parameters. The "correct" formula $^1/_2\cdot(3s^2-s)$ is found in the *Geometria incerti auctoris* [ed. Bubnov 1899: 346], in the agrimensor treatise of Epaphroditus and Vitruvius Rufus [ed. Bubnov 1899: 534], and in *Artis cuiuslibet comsummatio* [ed. Victor 1979: 158].

ragione è perché ogni tundo de qualunqua mesura se sia è intorno {intorno} 3 volte et $^1/_7$ quanto è el suo diametro, cioè el diricto de mezzo. Et per questa cagione ày a moltiplichare et partire como io t'ò dicto de sopra".

In general, in cases where mere copying mistakes and failing understanding can be excluded, the replacement of an error by a good solution is more likely to occur than the inverse process when a computational text is corrected. Comparison of these two problems might therefore seem to speak against the primacy of **V**:

V.11.4 Ancora diremo chosì. 7 libre di tornesi vagliono 9 libre de parigini. Che varrano 120 libre de tornesi? Fa così como de sopre abiamo dicto: 120 libre via 9 libre de parigini fanno 1080 libre de parigini; et parti per 7 libre de tornesi, cioè, parti 1080 in 7, che ne viene 154 libre et 5 soldi et 8 denari e $^4/_7$. Et cotanto diremo che vagliono le 120 libre de tornesi, cioè libre 154, soldi 5, denari $8^4/_7$ de parigini.

F.IV.3 Ancora diremo: 7 tornesi vagliono 9 parigini, dimmi quanto varranno le 120 di tornesi. Fa cosie. Die: poichè 7 tornesi vagliono 9 parigini, dunque 7 soldi di tornesi vagliono 9 soldi di parigini et 7 lire di tornesi vagliono lire 9 di parigini. Dunque multiplicha 9 via 120 lire de parigini, fanno lire 1080 e parti lire 1080 per 7, che nne viene lire 154 e soldi 8 e denari 6 e $^6/_7$. Et diremo che 100 lire di tornesi varranno lire 154 e soldi 8 et denari 6 et $^6/_7$, d'uno danaio et è fatta apunto.

Of the two incompatible answers, **V** provides the one that is "apunto", and the rule of thumb formulated above would therefore speak against the primacy of **V**.

However, rules of thumb *are* rules of thumb, and presuppose a particular context that they do not make explicit. Here, the presupposition – that one of the texts corrects the calculation of the other – is invalid. The originator of F does not *correct the calculation* of **V** but *the approach*, making specific translations at each of the levels *libre* and *soldi*. He therefore makes the calculation anew – and errs.[22] Probably because he presupposes his own method to be better, the *Verballhornung* seems not to worry him.

Evidently, all these arguments (and others of the same kind, which it

22 The error is $^1/_7$ of a *libra*, and thus goes back to an error of 1 *libra* before the division by 7 – 2 *libre* have been transferred to the level of *soldi* as 60 instead of 40.

would take up too much space to list) only show that **F** is the outcome of a rewriting of an original to which **V** is much more faithful[23] *in the chapters which are present in both versions.* They do *not* exclude that the chapters on algebra (V.16–19) and the final sequence of mixed problems (V.22) be additions.

However, other arguments speak strongly against this possibility. At first we may look at the contents of chapter **22**, which from the above general description seems to overlap chapters **14–15**. At closer inspection, however, the apparent overlap turns out to consist of duly cross-referenced variations and supplements; no single repetition can be found. This would hardly have been the case if a later hand had glued another problem collection onto an originally shorter treatise, given the frequency of wholesale borrowings of problems between different *abbaco* writings.

To this comes the homogeneity of **V**. On all levels – orthography, vocabulary, discourse, pedagogical style – the treatise is a seamless whole, also on points where it differs from **F** or other *abbaco* writings (e.g., the *io/tu* shift and the way diagrams are referred to). The same holds for the computational techniques when there is a choice, and for the mathematical approach (for instance the use of the partnership model, and the ever-recurrent emphasis that the approximate values for square roots *are* approximate). All this in itself does not necessarily mean that everything was written originally by the same author; but if it was not, it would have required a strong harmonization and reformulation by a later hand, as has

[23] If **V** were also the outcome of a process of rewriting, if would certainly contain passages in which inconsistencies betrayed dependency on something close to **F**. I have noticed none (apart from the corrupt **V.14.32**, which appears to mix up two numerically different versions of the same problem).

This does not mean that no inconsistencies can be found in **V** – but the only ones I have observed are shared with **F**, and point to the Arabic world. Firstly, a distinction is mentioned in the introduction (fol. 1^v) between "rocti sani e rocti in rocti", where "rocti in rocti" can hardly refer to anything but the *partes-de-partibus* usage of Arabic mathematics; but such "parts of parts" are never used afterwards. Similarly, the claim of chapter **8** that the prime numbers are the "most necessary" divisors is not born out by the calculations in the rest of the treatise; but it corresponds to a method that is prescribed by al-Qalasadi [ed. Souissi 1988: 42] when he discusses proportional sharing – namely to add all the shares and to resolve the sum (which is going to serve as divisor) into factors.

happened to the text as we find it in **F**. But a harmonization of this kind would also have affected the chapters that have a counterpart in **F**, and at some points we would certainly have found incongruities that betrayed the departure of **V** from an original stem which in these passages was closer to **F**. We must therefore conclude that whatever harmonization occurred to the text we find in **V** occurred *before* the text developing into **F** split off from the stem.

This still does not prove definitively that the chapters **16–19** and **22** were part of the treatise written by Jacopo in 1307. But their absence from **F** and **M** provides no evidence that they were not. As we shall see, at least the algebra must be dated well before Paolo Gherardi's work from 1328. All in all, the most reasonable assumption is thus that it was part of the original treatise.

Related algebraic writings

As we remember, one of the reasons that caused Van Egmond to assume a late date for the algebra of **V** was the presence of both third- and fourth-degree equations, where Paolo Gherardi's treatise from 1328 (henceforth **G**) has only third-degree equations. **V**, on the other hand, has only reducible cases, whereas Gherardi has irreducible cases as well; **V**, moreover, gives the rules without examples, while **G** illustrates all rules by examples. All in all, comparison combined with a tacit premise of cumulativity certainly does not speak unambiguously in favour of the primacy of **G**. But the very intricacy of the question suggests that closer investigation of the higher-degree problems may provide crucial information.

Two other texts can also profitably be compared with the sequence of higher-degree equations in **V** and **G**. One is an *abbaco* manuscript from Lucca from c. 1330 ([ed. Arrighi 1973], already referred to above), a conglomerate written by several hands. Its fols 80^v–81^v (pp. 194–197) contains a section on "le reghole dell'aligibra amichabile", which will be designated **L**; another section on "le reghole della chosa con asenpri" is found on fols 50^r–52^r (pp. 108–114; henceforth **C**). The other text is a *Trattato dell'Alcibra amuchabile* [ed. Simi 1994] known from a manuscript from c. 1365. The *Trattato* is part of Ricc. 2263 and is itself a composite consisting of three clearly separate treatises.[24] Of interest in the present connection is the second of these (fols 28ʳ–34ʳ), which I shall designate A.

Below follows a list of the "cases" present in these five works or chapters, with indication of the order. If a rule is present for the case, it is marked R if true, X if false; the presence of examples is indicated E, marked by subscript digits (E_{12} thus indicates that two examples are given; E_1 and E_2 in the same row but different columns indicate that examples differ, E_1 and $E_{1'}$ that they are identical apart from a numerical variation). The letters "p" and "n" indicate whether the division by which the equation is normalized is expressed as "partire per" or "partire in"; we shall see that this "neutral mutation" is an interesting parameter. *K* stand for *cubo*, C for *censo*, CC for *censo di censo*, *t* for *cosa*, *n* for *numero* (in whatever spellings the manuscripts use), and Greek letters for coefficients (implied by the reference to the plurals *cubi*, *censi*, and *cose*).

[24] The first of these (fols 24ʳ–26ʳ) carries the heading "Incomincia il primo trattato dell'alcibra amuchabile" and gives rules for the multiplication of signed entities and binomials. Fols 26ᵛ–27ᵛ treat of non-mathematical subjects. The second treatise, fols 28ʳ–34ʳ, starts without any heading, "Quando le cose sono iguali al numero". Fol. 34ᵛ is left blank, and the third treatise – fols 35ʳ–50ᵛ, a collection of problems – starts again without any heading, "Fammi di 10 due parti ...". In the first and third treatise we see the incipient transformation of syncopated into symbolic algebra; according to the edition, the second contains no traces even of syncopation.

Case	V	G	L	C	A
$\alpha t = n$	1.R,E_{12},n	1.R,E_{1}^{*},n	1.R,E_1,n	1.R,E_{1}^{*},p	1.R,E_{12},n
$\alpha C = n$	2.R,E_1,p	2.R,E_2,n	2.R,E_2,n	2.R,E_{2}^{*},n	2.R,E_1,p
$\alpha C = \beta t$	3.R,E_1,p	3.R,E_{1}^{*},n	3.R,E_{1}^{*},p	3.R,E_2,p	3.R,E_1,p
$\alpha C+\beta t = n$	4.R,E_{12},n	4.R,E_{1}^{*},n	4.R,E_{1}^{*},n	4.R,E_{1}^{**},n	4.R,E_{12},n
$\beta t = \alpha C+n$	5.R,E_{123},n	5.R,E_{2}^{*},n	5.R,E_{2}^{**},p	5.R,$E_{2}^{***¶}$,n	5.R,E_{123},n
$\alpha C = \beta t+n$	6.R,E_1,n	6.R,E_2,n	6.†	6.R,E_3,n	6.R,E_1,n
$\alpha K = n$	7.R,p	7.R,E_1,p	7.R,n	7.R,p	7.R,E_1,p
$\alpha K = \beta t$	8.R,p	9.R,E_1,p	8.R,n	8.R,p	8.R,E_1,p
$\alpha K = \beta C$	9.R,p	10.R,E_1,p	9.R,p	9.R,p	9.R,E_1,p
$\alpha K+\beta C = \gamma t$	10.R,n	15.R,E_1,n	10.R**,p	14.R,n	15.R,n
$\beta C = \alpha K+\gamma t$	11.R,n		11.R,n	15.R,n	16.R,n
$\alpha K+\gamma t = \beta C$					14.R,E_1,n
$\alpha K = \beta C+\gamma t$	12.R,n	11.R,E_1,n	12.R††,n	16.R,p	10.R,E_1,n
$\alpha K = \sqrt{n}$		8.R,E_1,p			11.R,E_1,p
$\alpha K = \beta t+n$		12.X,E_1,n			12.X,E_{1}^{*},n
$\alpha K = \beta C+n$		13.X,E_1,n			13.X,E_1,n
$\alpha K = \gamma t+\beta C+n$		14.X,E_1,n			
$\alpha CC = n$	13.R,n		13.R,p	11.R,p	17.R,n
$\alpha CC = \beta t$	14.R,p			12.R,p	18.R,p
$\alpha CC = \beta C$	15.R,p			13.R,p	19.R,p
$\alpha CC = K$	16.R,p			10.R,p	20.R,p
$\alpha CC+\beta K = \gamma C$	17.R,n				21.R,n
$\beta K = \alpha CC+\gamma C$	18.R,n				22.R,n
$\alpha CC = \beta K+\gamma C$	19.R,n				23.R,n
$\alpha CC+\beta C = n$	20.R,n				24.R,n
$\alpha CC = n$	7.R,n				
$\alpha CC = \beta t$	8.R,p				
$\alpha CC = \beta C$	9.R,p				
$\alpha CC = \beta K$	10.R,p				
$\alpha CC+\beta K = \gamma C$	11.R,n				
$\beta K = \alpha CC+\gamma C$	12.R,n				
$\alpha CC = \beta K+\gamma C$	13.R,n				
$\alpha CC+\beta C = n$	14.R,n				

* The statement has a lacuna, and should read "Trouami 2 numeri che tale parte sia l'uno dell'altro come 2 di 3 e, multiprichato il primo per se medesimo et poi ⟨per⟩ quello numero faccia tanto quanto e più 12".

† Absent; but since the ensuing text refers to "6 reghole", this is clearly by involuntary omission.

** The rule should read "Quando li chubi ⟨e li censi⟩ sono egualj alle cose [...]".

†† The rule should read "Quando li chubi sono egualj ⟨a' censi⟩ e alle chose [...]".

¶ In the short collection of further illustrative examples, C also has the problem E_1 of V.

The first obvious observation is that the distribution of divisions *per* versus divisions *in* is strikingly similar in all cases. To some extent this *may* be determined by the subject-matter – for some obscure reason (if not by pure accident), normalizations *per* mostly occur when only two powers are involved. But this does not explain all, and the agreement still shows that the five texts are closely related.

Most kindred are **V** and **A**, where the agreement on this account in complete in all shared cases. There is also full agreement in the examples that illustrate these (and it should be observed that the examples are not formulated in terms of *censi* and *cose*, and only to a limited extent in purely numerical terms ("Trovami 2 numeri che siano in propositione sì como è 4 de 9. Et multiprichato l'uno contra l'altro faccia quanto ragionti inseme",[25] and the like). There is no doubt that one depends closely on the other, or on a very near parent – the only question concerns priority.

The passage that most clearly shows that **A** is derived from a predecessor of **V** is found in the example E_2 for the case $\alpha C+\beta t = n$ (**V.16**.10, fol. 39ʳ; **A**, fols 29ᵛ–30ʳ, p. 25*f*). In **V** we find

> Et però di': multiprichare radice de 54 meno 2 via radice de 54 meno 2. Et cotanto varrà el censo. Che in verità, radice de 54 meno 2 via radice de 54 meno 2, fa 58 meno radice de [26] et abbiamo che vale el censo 58 meno radice . Et noi ponemo avesse el primo uno censo. Dunqua vene ad avere 58 meno radice de . ⟨Ora sappi el secondo, che ponesti ch'avesse $^1/_4$ censo e $17^1/_2$ numeri. Adunqua piglia el $^1/_4$ de 58 meno radice de 864⟩ ch'è $14^1/_2$ meno radice de 54, sopra el quale vi giongi $17^1/_2$; fanno 32 mino la radice de 54. Et così abiamo che el primo à 58 meno la radice de . Et el secondo homo à 32 meno radice de 54.

The corresponding passage in **A** runs as follows:

> Abiamo che vale la cosa radicie di 54 meno e 2; vie radicie di 54 meno 2, e cotanto varrà il cienso, che 'n verità fa 58 meno radicie di 864. E nnoi ponemo che 'l primo avesse uno cienso. Dunque avrà il primo 58 meno radicie di 864. Ora sappi il secondo, che ponesti ch'avesse $^1/_4$ cienso e $17^1/_2$ numeri. Adunque piglia il $^1/_4$ di 58 meno radicie di 864, ch'è 14 e $^1/_2$ meno radicie di 54, sopra il quale giungni $17^1/_2$, fanno 32 meno la radicie di 54. E così abiamo che 'l

[25] **V.16**.7 (fol. 37ᵛ).

[26] Instead of «864», the ms leaves open c. 2 cm. In the margin the copyist writes the commentary «così stava nel'originale spatii».

primo à 58 meno radicie di 864 e'l secondo huomo à 32 meno radicie di 54.

The passage in ⟨ ⟩ in **V** is filled out according to the words of **A**, but in the orthography of **V**. That it fits perfectly into the rest only confirms that the two texts are close to each other. The empty spaces in **V** are more informative. They demonstrate that **V** descends via attempted faithful (though, as we see, not always actually faithful) copying from a prototype (henceforth **V′**) prepared in the original process of computation (at least in as far as this problem is concerned) – a manuscript where the author left open the spaces where he might insert the result when he had calculated 16×54, but then forgot to do so. **A**, instead, either derives from **V′** via an intermediate manuscript **A′** where the calculations were performed – or they are made directly in **A**. In any case, **V** is not derived from neither **A** nor the hypothetical **A′**.[27]

A is also a witness of other more fundamental innovations. Firstly, it introduces examples for several of the higher-degree cases, all of which appear in **V** "senza niuna dispositione", "without any exposition", that is, without illustrative examples. These are all facile in construction, in contrast to many of those that illustrate the second-degree cases. As an example we may look at the illustration of the case "li chubi sono iguali a' ciensi ed alle cose" (fol. 32ᵛ, p. 30):

[27] Comparison of other parallel but diverging passages lead to the same conclusion – for instance the remarks that close the discussion of the double solution of the case $\beta t = \alpha C+n$. In **V.16.14** (fol. 40ᵛ) they run

> Et abi a mente questa regola. In bona verità vorrebbe una grande despositione; ma non mi distendo troppo però che me pare stendere et scrivere in vile cosa; ma questo baste qui et in più dire sopra ciò non mi vo' stendere.

To this correspond in **A** (fol. 31ʳ, p. 27) the following inconsistent passage (no reader was ever asked to "keep in mind" that the author should have given a more thorough explanation):

> Ed abi a mente che questa reghola vorebe una grande dispusitione, ma non mi ci distendo tropo che melo pare scrivere multa cosa e questo basti.

A, or its original, as we see, uses (some precursor of) **V**, and squeezes two sentences into one – that **V** should be able to take a single meaningless period from some **A′** and expand it meaningfully into two as here is quite unlikely). And whereas the author of **V′** is tempted to elaborate the argument, the writer of **A** or its original finds his source too loquacious.

Trouami 3 numeri che sieno in proporzione insieme come 3 di 4 e come 4 di 5 e, multiprichato lo primo per se medesimo e poi per lo numero, faccia tanto come lo secondo multiprichato per se medesimo e posto in suso lo terzo numero.

The first number is taken to be "3 cose" and the others "4 cose" and "5 cose", respectively, which yields the equation

$$(3 \text{ cose})^3 = (4 \text{ cose})^2+(5 \text{ cose}) .$$

Since no care is taken to avoid irrational solutions, it is obviously easy to construct suitable illustrations for all cases.

We also find some additional cases – not only $\alpha K = \sqrt{n}$ but also the two irreducible cases $\alpha K = \beta t+n$ and $\alpha K = \beta C+n$, both solved by means of the wrong formula $t = \sqrt{\frac{n}{\alpha}+(\frac{\beta}{2\alpha})^2}+\frac{\beta}{2\alpha}$, and illustrated by means of examples that lead to unhandy irrational solutions that do not invite to verification.

The same 3 additional cases are found in **G**, with the same rules and the same examples. In **G**, we notice, *all* rules are provided with illustrations, and in all third-degree cases where **A** has an example it recurs in **G**. Again we have to ask whether **A** is derived from **G** (or some **G′** close to **G**) or vice versa.

The treatment of the case $\beta t = \alpha C+n$ solves this problem. **G** has only one of the three illustrations of this case that are found in both **V** and **A**,– in a numerical variation E_2.. that leads to an irrational solution. **V** and **A**, instead, share a nice integer solution – or rather two, because they have and explain the existence of a double solution, which **G** ignores though having it in the rule. Moreover, as we have seen in note 27, the explanations given in **V** and **A** are closely related and not independent discoveries of the same mathematical fact. No doubt, therefore, that Paolo Gherardi borrows from a predecessor **A′** of **A** in which the innovations with respect to **V′** were already present.

We notice that **A** has examples of the rules until a certain point, and then rules only. **G**, as observed above, has examples to all rules, but with one exception, all its cases are also in **A**, that is, already in **A′**. The exception is the case $\alpha K = \gamma t+\beta C+n$, which is solved as if it had been $\alpha K = (\gamma+n)t+\beta C$ (or, with the same wrong formula, as $\alpha K = \beta C+(\gamma+n)$). This is a fallacy so to speak of a higher order than the others, and we may assume that it was added by Paolo Gherardi himself or somewhere between **A'** and **G**.

Next we shall have to look at **L** and **C**. None of them have examples for the higher-degree cases, in which respect they are closer to **V** than **A**. **C** is the one that comes closest to **V** and **A** in the distribution of divisions *per* respectively *in*, and also the one that has the largest number of rules (but like **L** none that are not found in both **V** and **A**). However, there are some differences in the *per*/*in* distribution, and some of the illustrative examples for the second-degree cases are also different. Given the complete agreement between **V** and **A** on these and many other accounts, **A** cannot descend from neither **L** nor **C**; they will have diverged from the line connecting **V'** and **A'**. Since **L** mostly differs from **A** on the points where **C** differs and has none of the rules which are omitted in **C**, they are likely to derive from a common precursor **C'**.

All in all, we are led to the stemma shown on the following page for the five expositions of algebra (**F** and **M**, containing no algebra, do not enter).[28] **V''** designates the manuscript in which the list of silver coins

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[28] Further arguments for this stemma follow from comparison of the precise words of the different texts. As an illustration, we may first quote the rule for the case $\alpha K = \beta C+\gamma t$:

V: dei partire ⟨ne⟩li chubi et poi dimezzare li censi, et multiplichare per se medesimo et agiungere alle cose, et la radice dela summa più el dimezzamento de' censi vale la cosa.

L: si vuole partire ne' chubij e poj dimeçare ciensi e multiprichare per se medesmo e giugnere sopra alle chose, e lla radicie della somma più il diççamento de' censi vale la cosa.

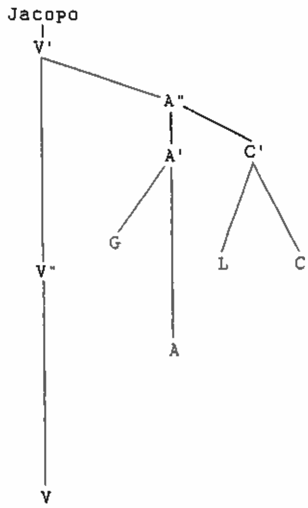
C: dovemo partire per li chubi e poj dimeççare i censi e multiprichare per se medesmo e porre sopra le chose, acciò che nne viene sarae radicie di quello e più lo dimeççato de' censi, e chotanto varrae la cosa.

A: dei partire ne' chubi, poi dimezare i ciensi e multiprichare per se medesimo e giungniere alle cose e la radicie della somma più il dimezamento de' ciensi vale la cosa.

G: dovemo partire ne chubi e poi dimezare le cose [*sic*, read "censi"] e multiplicare questo dimezamento per se medesimo e porre sopra le cose, e ℞ di quella somma che fara piue lo dimezamento varra la cosa.

A, we note, is very close to **V**. **L**, **C** and **G** diverge in different directions although with some similarity between **C** and **G**.

was first displaced, and whose explanation of this was copied in **V**; the thick line signifies (attempted) faithful copying, thin lines more or less creative use and re-elaboration.



Since Jacopo wrote in 1307 and Paolo Gherardi (**G**) in 1228, and since **C** and **L** are dated around 1330, **A'** and *a fortiori* **A"**, the point where divergence toward **C** and **L** begins, have to fall before that date; this constitutes no absolute proof that **V'** – the point where the algebra got into Jacopo's treatise and was harmonized stylistically with the rest – is the original writing of the treatise. But so little time is left, allowing a reasonable distance between the certainly different points **A"**, **A'** and **G**, that it is the only reasonable assumption, which I shall therefore adopt in what follows. If I am mistaken, it is a least certain that **V'** has to be located very soon after 1307, and well before 1328.

The case $\alpha C = \beta t{+}n$ (forgotten in **L**) is treated in this way:

V: se vole partire nelli censi, et poi dimezzare le cose, et multiplicare per se medesmo et giongere al numero. Et la radice dela summa più el dimezzamento dele cose vale la cosa.

C: dovemo partire ne' censi, e poj dimeççare le cose e multiprichare per se medessmo e pollo sopra il numero, e sarae radice di quello e piue lo dimeçamento delle cose; e cotanto vale la cosa.

A: dei partire ne' ciensi e poi dimezare le cose e multiprichare per se medesimo e giungniere al numero; e la radicie della somma più il dimezamento delle cose vale la cosa.

G: de partire ne censi e poi dimezare le cose e multipricare per se medesmio a ragiungnere sopra lo numero, e la radice di quello piue lo dimezamento dele cose vale la cosa.

Here, **A** shares one of the three changes from **V** to **G** (*se vole* > *dei*, but neither *giongere al* > *ragiungnere sopra* nor *dela summa* > *di quello*). This time, **G** and **C** diverge in totally different ways.

Jacopo's "innovations"

I am publishing an edition and translation of the algebraic chapters of **V** elsewhere [Høyrup 1998a], for which reason I shall only summarize their characteristics here.

Globally, it is remarkable that Jacopo's algebra does not share a single example *and only a single rule* with the Latin algebras – that is, with the sum-total of Robert of Chester's and Gherardo da Cremona's translations of al-Khwārizmī and the revision perhaps to be ascribed to Guglielmus de Lunis; the translation of Abū Kāmil; and Leonardo Fibonacci's *Liber abbaci* and *Pratica geometrie*.[29]

It may amaze that only a single rule – namely for the case $\alpha t = n$ – is shared with the Latin works. The explanation is that the rules of the latter for the second-degree cases presuppose problems to be normalized[30] (the first-degree problem is evidently non-normalized – if it were not, the statement of the problem would already *be* the solution). Jacopo, as the other treatises discussed above, presupposes problems to be non-normalized, for which reason the first step of all rules is a normalization.

The Latin algebras (with the exception of the *Pratica geometrie*, whose algebra is subordinate and not the subject-matter proper) illustrate the rules by examples formulated in terms of the representation – "the *census* and 10 roots are made equal to 39", etc. None of Jacopo's problems are of this sort (nor are those of **A**, **C**, **L** and **G**). Some of Jacopo's (and all of the higher-degree problems of the others) are pure-number problems ("find two numbers which are in the same proportion as ...", etc.); but others are dressed as real-life problems concerned with partnerships, commercial profit from travels, etc. One of these, strikingly, deals with the square root of an amount of real money.[31] This problem type is evidently the origin

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[29] Ed., respectively, [Hughes 1989], [Hughes 1986], [Kaunzner 1986], [Sesiano 1993], [Boncompagni 1857a] and [Boncompagni 1862].

[30] Many of their examples are certainly non-normalized, and then the texts tell how to reduced them to normalized form; but the explicit *rule* only applies when this form has already been brought about.

[31] "E sonno due homini che ànno denari. Dice el primo al secondo: Se tu me dessi 14 de toi denari, che io li racchozzasse co' mey, io arei 4 cotanti de te. Dice el secondo al primo: se tu me desse la radice de toy denari, io arei denari 30" (**16.10**,

of the Arabic *māl-jiḏr*-techniques. With al-Khwārizmī and Abū Kāmil, however, it only survives as the standard representation, and for this reason it is also absent from the Latin algebras (which translate *māl* correctly as *census* or *substantia*, but give no hint that these terms should be understood literally).[32]

As most fourteenth-century *abbaco* algebras, Jacopo's contains no geometric proofs (nor any other argument) for the validity of its rules. Even this is of course in contrast to all the Latin algebras.[33]

Already Karpinski observed that Jacopo's order of the first six cases differs from the traditional al-Khwārizmīan order, which is also the order of Abū Kāmil's *Algebra*:

$$(1)\ C = \beta t,\ (2)\ C = n,\ (3)\ \alpha t = n,\ (4)\ C+\alpha t = n,\ (5)\ C+n = \beta t,\ (6)\ \beta t+n = C$$

The order of the *Liber abbaci* is different: 1–2–3–4–6–5. That of Jacopo (and of **A**, **G**, **L** and **C**) is also different: 3–2–1–4–5–6.

A final point on which Jacopo's algebra differs strikingly from the Latin translations is of course the *systematic* inclusion of reducible higher-degree cases (both Abū Kāmil and the *Liber abbaci* comprise bi-quadratic problems, but they offer no systematic treatment).

There is still a tacit tendency within the historiography of mathematics to identify Arabic algebra exclusively with what can be found in the works belonging to the "high" tradition: al-Khwārizmī, Thābit ibn Qurrah, Abū Kāmil, al-Karajī's *Fakhri*, al-Khayyāmī – and in practice with al-Khwārizmī

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fol. 39r).

32 Neither al-Khwārizmī nor Abū Kāmil have forgotten completely the original role of the *māl* as basic unknown – the former after having found the root goes on to determine the *māl*, the latter invents a geometric procedure from which the *māl* follows directly. However, they both present the case *māl* made equal to number in normalized form (cf. imminently), that is, state the value of the *māl* directly; their own view must thus be that *the root* is the fundamental variable.

33 I disregard the brief presentation of "gleba mutabilia" in *Liber Alchorizmi de pratica arismetice* [ed. Boncompagni 1857b: 112*f*]. The algebra section is not in Allard's partial edition of the *Liber Alchorizmi* [1992] but present in manuscripts that are as far as possible from each other in the stemma – see [Høyrup 1998c: 16 n.7] and thus doubtless part of the original work and no interpolation. But it appears to have had no impact whatsoever.

and Abū Kāmil alone. On this background, the presentation of the subject in **V** is so different from "Arabic algebra" and so innovative that the doubt as to its ascription to an otherwise unknown writer from 1307 becomes understandable – not least because it avoids all the erroneous rules that flourish in most *abbaco* algebras from Paolo Gherardi to Piero della Francesca.

But this identification is false, and Jacopo's apparent innovations can also be traced in the Arabic world.

Let us first look at the order of six basic cases. The order 1–2–3–4–5–6 is not only the order of al-Khwārizmī and Abū Kāmil (Arabic as well as Latin translations); it is also the order of Thābit ibn Qurrah's *Verification of the Problems of Algebra through Geometrical Demonstrations* [ed., trans. Luckey 1941: 105–107] – only 4–5–6); ibn al-Bannā'''s *Talkhīṣ* [ed., trans. Souissi 1969: 92]; ibn al-Yāsamīn's *Urjuza fi'l-jabr wa'l-muqābalah* (paraphrase in symbols in [Souissi 1983: 220–223]); and ibn Turk [ed. Sayılı 1962: 145–152] (1–4–5–6 only). It is certainly to be regarded as the classical order.

Al-Karajī, however, has the sequence 3–1–2–4–5–6, both in the *Kāfī* [ed., trans. Hochheim 1878: III, 10–13] and in the *Fakhrī* [paraphrase Woepcke 1853: 64–71]; he is followed by al-Samaw'al, al-Kāšī, and by Bahā' al-Dīn [ed., trans. Nesselmann 1843: 41*ff*]. Finally, Jacopo's arrangement is told in al-Māridīnī's commentary to ibn al-Yāsamīn's *'Urjūza* from c. 1500 [Souissi 1983: 220] to be what is used in "the East", and it is indeed the order of al-Miṣṣīṣī, al-Bīrūnī, al-Khayyāmī and Šaraf al-Dīn al-Ṭūsī [Djebbar 1981: 60]; but it is also followed by al-Qurašī (thirteenth-century, born in Andalusia, active in Bugia in Algeria) [Djebbar 1988: 107]. It thus appears to have been quite widespread in Jacopo's epoch.

Widespread in early second-millennium Arabic algebra was also the solution of reducible higher-degree equations like Jacopo's cases 7–14;[34] however, apart from the solutions by means of solid geometry due to al-Khayyāmī and others, Arabic algebra did not treat the irreducible cases; Van Egmond is certainly right when asserting [1978: 163] that "no Arabic algebraist could have written a treatise so full of elementary errors" as **G**, and that "Leonardo Pisano was far too intelligent to have written such

[34] See al-Karajī's *Fakhrī* [paraphrase Woepcke 1853: 71*f*] and ibn al-Bannā'''s *Talkhīṣ* [ed., trans. Souissi 1969: 96] – and in general [Djebbar 1981: 107*f* and *passim*].

foolishness".

The question of normalization is intricate. As pointed out, the second-degree cases are all normalized in the Latin algebras; so are the rules which Thābit and ibn Turk quote in their proofs.[35] In the published Arabic text of al-Khwārizmī [ed. Mušarrafa & Aḥmad 1939], in contrast, they are non-normalized.

At close analysis, however, this Arabic text turns out to have been submitted to at least three revisions since the stage of which Gherardo's translation is a witness (see [Høyrup 1998b]). Given Gherardo's grammatical faithfulness we may be quite confident that even al-Khwārizmī's cases were normalized; the extant text is a witness of the process in which living, practised algebra in the Arabic world drifted toward inclusion of the normalization step in the explicit rule.

Al-Karajī's treatises show us the same process at work. The *Kāfī* [trans. Hochheim 1878: III] only gives rules for the three simple cases, but all of these are for the non-normalized form.[36] According to Woepcke's paraphrase of the *Fakhrī* [1853: 64*ff*], the same principle prevails here.

The *Kāfī* exhibits other interesting features. First of all, it gives no geometric proofs; this characteristic recurs in the Maghreb tradition, and with Bahā' al-Dīn. Secondly, its use of the verbs *jabara* and *qabila* (habitually translated "restore" and "oppose" in the context of algebra, and the very verbs which in nominalized form give the technique its name, *Al-jabr wa'l-muqābalah*) differs from the usage established by al-Khwārizmī[37] and seems to reflect pre-al-Khwārizmīan ways of speaking[38], as indeed

35 Ed., respectively, [Luckey 1941: 110-112] and [Sayılı 1962: 144–153].

36 The composite cases are represented by examples only and without a general formulation.

37 In this usage, "restoration" means addition to both sides of an equation, which cancels a subtractive member; "opposition" means subtraction from both sides.

38 Cf. [Saliba 1972]. According to the *Kāfī* [trans. Hochheim 1878: III, 10], *al-jabr* is the elimination both of factors and of subtracted members (added members are not mentioned explicitly), which leads to *al-muqābalah*, "opposition" as two sides of a reduced equation. The *Fakhrī* has the same usage, but includes explicitly the elimination of added terms under *al-jabr* – cf. quotation and commentary in [Woepcke 1853: 64].

confirmed by Abū Bakr's *Liber mensurationum* [Høyrup 1996: 50f],– and if we go to Jacopo we discover that he uses *ristorare* in the same way.[39]

Irrespective of the level which he attains in the *Fakhrī*[40], this terminological peculiarity of al-Karajī's works show that his starting point is the "low" or "non-scholarly" current to which also a Bahā' al-Dīn belongs – the current which had never felt the need to provide "Greek-style" proofs for its algorithms, and which (with all the provisos that this notion may ask for) came to dominate the mathematics of the *madrasah* in the era when scientific activity was "naturalized" in Islam [Sabra 1987]. This link of the *Kāfī* to the "low" tradition is confirmed by its practical geometry – see [Høyrup 1997, *passim*].

Every characteristic of Jacopo's algebra so far discussed thus points to the "low", non-Grecisized register of Arabic algebra. The only possible trace of a tie to Gherardo's translation or to Fibonacci is indeed of terminological character: the choice of *census*, transformed into *censo*, as the equivalent of Arabic *māl*. But even this trace is not fully unambiguous. The same choice was indeed made in the non-algebraic *Liber augmenti et diminutionis* [ed. Libri 1838: I, 304–371], and it may thus have been more routinely than it seems to us.

Nonetheless, we cannot exclude that Jacopo's use of *censo* derive from the Gherardo-Fibonacci-tradition. But his apparent innovations are clearly not innovations within this tradition – not primarily because they are borrowed (as they clearly are) and not innovations but because they do not fall "within this tradition". Instead, they – and Jacopo's whole treatment of algebra – constitute the earliest evidence of the establishment of a *new* algebraic tradition, borrowing once again from the Arabic world – not, however, from the classical treatises as the Latin works had done, but from the living "low" tradition (which was certainly also exploited by the all-

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39 One example of the non-al-Khwārizmīan use (among several) is **V.16**.13 (fol. 40^r), "Ristora ciascheuna parte, cioè de cavare 24 cose de ciascheuna parte". I suspect that Jacopo's "equation" (*raoguaglamento*, 16.13, fol. 40^v) may correspond to *al-muqābalah*.

40 "Low" and "high", indeed, do not refer to any measure of quality but to social prestige – and, in the actual case, to the social prestige which *our* world ascribes to various types of scientific activity.

devouring Fibonacci, but not as his sole resource).

But we may say more. If we compare Jacopo's explanatory style (as exemplified by the remarks (a)–(h) above, p. 7) with other *abbaco* writings, it becomes obvious that he is conscious of having a particular job to do, and that his work is indeed not only an accidental first extant witness of a new tradition but the very establishment of that tradition: the Tuscan *abbaco* with algebra.[41]

As we know, his treatise is not the first *abbaco*. It is younger than a late thirteenth-century Umbrian specimen drawn from Fibonacci's *Liber abbaci* [ed. Arrighi 1989], which provides us with evidence that an environment had already emerged which felt a need for the new genre. We may presume that Jacopo was deliberately catering for this need, that he deemed insufficient what was already at hand – better, that he "recognized" this insufficiency: the immediate use of his work by others shows that the supposition was justified. This aim is well reflected in the introduction to the treatise, which first states that "el senne è el più nobile thesoro che sia al mondo" and next goes on in the same vein (in high-flown terms that are not borrowed from the scolarly tradition) for a whole page.

In any case he chose to draw on material borrowed from somewhere in order to put together his *Tractatus algorismi*, as he calls it in the *incipit*. The question is, *from where*?

Ultimately, of course, the material comes from the Arabic world. But there is no single Arabism in the work (unless we count *fondaco*, "warehouse", derived from Arabic *funduq*; but this is no mathematical term and will have been naturalized in commercial life before being used in mathematical problems). Jacopo must therefore be assumed to have drawn on a tradition that was already well established in Romance language.

Since the *Algorism* was written in Montpellier, the immediate source can be presumed to be Provence. This is supported by the settings of

[41] This being really *first* agrees well with the lack of a standardized terminology which was noticed on p. 5. Standardization results from honing and trimming and is difficult to achieve in the first instance (and not very relevant when those for whom texts are produced are not themselves familiar with the standardized terminological canon – in order to make such a public grasp what is meant, variation is more adequate).

problems. Rome and Montpellier, together with Florence, Bologna, Avignon, Toulouse, "overseas", Genova, Aigues-Mortes, and Lucca, constitute the horizon of travelling; Florence and Montpellier turn up twice. Other locations, from Paris and Nîmes to Sicily and Alexandria, are only mentioned as domiciles of their currency and measures.

Apart from "overseas", this commercial horizon does not reach the Arabic world. We know, on the other hand, that the Occitan-Catalan area was largely a cultural unity (Paolo Gherardi's commercial horizon encompasses Barcelona as well as Mallorca), and that Catalan trade was mainly directed toward the Arabic world in the second half of the thirteenth century [Abulafia 1985]. The only reasonable assumption seems to be that a tradition for practical reckoning was already established in this Occitan-Catalan orbit around 1300, most likely somewhat earlier, and that Jacopo working (and learning) here decided to spread the gospel. He was not the only Florentine to go here; in 1328, Paolo Gherardi was to be found in Montpellier – and in the meantime it is a good guess that **V′**, **A″** and **A′** had been produced in the same locality.[42]

This turns a commonly assumed cause-and-effect arrow around: Occitan practical mathematics as known in particular from the fifteenth-century *Algorism* (same name!) from Pamiers[43], and indirectly from Chuquet's *Triparty*, is not primarily derived from Italian *abbaco* mathematics but rather (with ample space for secondary mutual interactions) its root. As Gherardo da Cremona and other scholars would go abroad in the twelfth century in order to get hold of that knowledge for which the schools of their Latin world had discovered an urgent need, so Tuscan masters would go to Montpellier in order to acquire what was needed (for practical use, or culturally) in *their* towns. Whether they stayed like Gherardo "until the end of life" engaged in that quest for knowledge which Jacopo asserted

[42] Only linguistic analysis of **M** may tell whether the common precursor of this manuscript and **F** is also likely to have been produced here. The fact that **M** was produced in Genova supports the assumption, but the orthography of **F** (which may of course characterize only **F**) seems very Tuscan. (But see below, p. 30).

[43] See the partial edition in [Sesiano 1984]. "Algorism" remained the standard name for the type in French and Occitan area – see the *incipit* of Mathieu de Préhoude's treatise as quoted in [Cassinet 1993: 253].

to be more beautiful than anything else or returned like Adelard to pursue a career we shall probably never know.

Oblique light on the Arabic "low" tradition

"Low" traditions tend to have a low prestige even in the historiography of mathematics. Also for this reason (others exist), we tend to be much better informed about Archimedes Arabus than about the "low" register of Arabic mathematics.

Paradoxically as it may seem, Jacopo's treatise may therefore also inform us about aspects of Arabic mathematics that we know little or nothing about.

I shall concentrate on three points – first on the problem type that involves the square root of an amount of money (see note 31). Such problems (involving the square root of some real countable entity) are well known from India, not least from Mahāvīra's *Ganita-sāra-sangraha* [ed., trans. Raṅgācārya 1912], and they are obviously the origin of the *māl-jiḏr* (possession and square-root of possession) problems that are the exemplars of Arabic *al-jabr* (as also reflected in the fact that the number to which for instance the sum of possession and roots is equal is a number of dinars). But as free-running problems I do not remember to have encountered them in Arabic material.

On the other hand, direct links from Montpellier or Catalonia to India can be safely disregarded. Since the problem type is not one which is likely to be invented spontaneously, we may thus conclude that it *was* present in the Arabic Mediterranean; and we may start looking for it.

My next point is the second-order approximation to irrational square roots. The choice in **V** (see p. 11) is of course based on a fallacy that might turn up everywhere; but the attempted improvement in **F** can hardly be explained as anything but a mistaken use of the correct formula, and even this must therefore have been around, if not in Montpellier then in an environment with which the Occitan-Catalan masters were in contact (which, conversely, provides us with a supplementary argument that the precursor of **F** and **M** will have been produced in the Occitan rather than the peninsularian Italian area, cf. note 42).

The correct formula, as pointed out in note 20, is known from ibn al-

Bannā' and al-Qalasādī (but only the explanation of the latter makes us sure what the former means). Once again, the Jacopo manuscripts inform us that it must have circulated more widely.

My third and final point regards the problems in **V.19**. They all deal with somebody who directs a *fondaco* for a number of years, and it appears to go without saying that he receives his yearly salary in geometric progression. The statements can be translated as follows (s_i denotes the salary of the i'th year, N the total number of years):

19.1: $N = 3$, $s_1+s_3 = 20$, $s_2 = 8$. Uses $s_1 \cdot s_3 = s_2^2 = 64$.
19.2: $N = 4$, $s_1 = 15$, $s_4 = 60$; finds the quotient as $\sqrt[3]{(s_4/s_1)}$.
19.3: $N = 4$, $s_1+s_4 = 90$, $s_2+s_3 = 60$. Uses $s_1 \cdot s_4 = s_2 \cdot s_3 = \frac{(s_2+s_3)^3}{3(s_2+s_3)+(s_1+s_4)}$.
19.4: $N = 4$, $s_1+s_3 = 20$, $s_2+s_4 = 30$. Finds the quotient as $(s_2+s_4)/(s_1+s_3)$.

None of the problems employ the *censo-radice* technique; in the usage of C, they are solved "without a rule". One may suspect that the refined solution of **19.3** (and probably also **19.4**) draws upon that polynomial algebra which was already a concern of the Maghreb school.[44] Apart from that we notice that both **19.1** and **19.3** are reduced to the problem $a+b = \alpha$, $a \cdot b = \beta$. In all cases this problem is solved by the method of "average and deviation" known both from *Elements* II.5, from Old Babylonian "algebra" and from Abū Bakr's *Liber mensurationum*.

I am not aware that this technique is used in known Arabic sources for anything but the kind of geometric riddles that we find with Abū Bakr (and elsewhere, both in the *Kāfī* and in ibn Thabāt's *Ghunyat al-Ḥussāb* [ed. Rebstock 1993]). But Jacopo provides us with fairly certain evidence that it *was* used more generally as an algebraic technique (the coupling to an isolated instance of complicated polynomial algebra seems to exclude a Catalan-Occitan invention).

[44] Another possibility, as pointed out by Enrico Giusti and Lucia Grugnetti, is the use of the arithmetical books of the *Elements*. If so, however, this origin was well forgotten: the principle that $s_1 \cdot s_3 = s_2^2$ (etc.) is stated explicitly as a principle time and again – but none of the other theorems that would have to be used are explained or referred to in any way.

Reception and transformation

I shall close with some sketchy remarks on the impact of Jacopo's work on the further development of the *abbaco* tradition, in particular in the domain of algebra.

Most conspicuous is the reception and transformation of the higher-degree cases. Quite soon, on one hand (namely in **A'** though not yet in **A"**), examples were constructed which might illustrate the use of the rules; the extreme uniformity of these examples and their conspicuous *ad hoc* character leave little doubt that they were new creations, modelled after Jacopo's illustrations of the case $\alpha c = n$ (**V.16**.5, fol. 37ᵛ),

> Trovame doi numeri che siano in propositione sì come è 2 de 3; et multiplicato ciascheuno per se medesimo, et tracta l'una multiplicatione dell'altra, remangha 20

and of the case $\alpha C = \beta t$ (**V.16**.7, fol. 37ᵛ),

> Trovami 2 numeri che siano in propositione sì como è 4 de 9. Et multiprichato l'uno contra l'altro faccia quanto ragionti inseme

or after some similar model. Some of these new illustrations will have been in **A'**, others turn up only in G and later treatises.

On the other hand, Jacopo's list of reducible and thus correctly solved cases was soon extended so as to comprise also irreducible (and wrongly solved) cases. Even this process can be see to have begun in **A'**, but was to go much further within short.[45]

The reason for this development – absurd as it is if we assume mathematics and "mathematicians" to strive after Platonic truths – is probably the competition between *abbaco* masters. These, in order to get appointments, students or fame, had to show off. When able to accomplish this by means of correct rules they would do so; when not they would pay in counterfeit coin.

Counterfeit coin, of course, only serves when believed to be genuine. The reason that the wrong rules would serve in the game was another development that had its starting point in Jacopo's algebra but was allowed to run wild (most likely because it served the purpose). Already some of

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[45] See [Van Egmond 1983] on the 198 cases treated in Dardi da Pisa's *Aliabraa argibra*, probably from 1344.

Jacopo's examples lead to irrational solutions: **V.16**.9, fol. 38ᵛ; **V.16**.10, fol. 39ʳ; **V.16**.12, fol. 39ᵛ; and the second *fondaco* problem, **V.19**.2, fol. 44ʳ). In the first and third of these instances Jacopo points out the mishap (in the first thus "[...] fanno 600. Trova la sua radice, la quale è sorda, cioè, che è manifisto, de non avere radice appunto"); the second instance is the case where he leaves an open space in the manuscript when unable to transform directly $4\sqrt{58}$ into $\sqrt{864}$. In none of the cases does he give an approximate value for the root.

But the other problems have nice solutions, and thus show that they are constructed backwards from these – for instance **V.16**.13 fol. 39ᵛ,

> Uno fa doi viaggi, et al primo viagio guadagna 12. Et al secondo viagio guadagna a quella medesema ragione che fece nel primo. Et quando che compiuti li soi viaggi et egli se trovò tra guadagniati et capitale 54

which has the solutions 6 and 24.

C finds an easy way to display originality without taking any risks, halving both given numbers; **L** keeps 12 but changes 54 into 64, which gives the solutions 4 and 36. **G** is more radical, keeps 12, but changes 54 into a nice 100. The price to be paid is that the solution (**G** gives only one) is $\sqrt{31\frac{1}{4}}+2\frac{1}{2}$. Obviously, the solution was not the starting point.

Nor is the solution planned in advance in the newly created examples for the higher-degree cases. In consequence, the irreducible problem

$$(2t)^3 = (2t+3t)+16$$

has a solution $t = \sqrt{2\frac{25}{256}}+\frac{5}{16}$, which certainly does not lend itself to easy control – and, in further consequence, such rules and solutions could be copied by others who would not even recognize that they were trading in mathematical nonsense.

If we turn our interest to the treatment of the second-degree cases, we shall discover that two of the four illustrative examples that are found in **A**, **G**, **L** or **C** but not in Jacopo's chapter **16** (namely E_2 of the case $\alpha C = n$ and E_2 of the case $\alpha C = \beta t$) are slight variations of Jacopo's **22**.7 (fol. 52ᵛ),

> Trova uno numero che, tractone el $^1/_2$ e'l $^1/_4$ e'l $^1/_6$, e lo remanente multiplicato

per se medesimo, faccia quello medesimo numero.

In C [ed. Arrighi 1973: 108*f*], these two "new" examples are formulated

> Truovamj uno numero che, trattone il $^1/_3$ e 'l $^1/_4$, lo rimanente multiprichato per se medesmo faccia 20

and

> Truovamj uno numero che, trattone il $^1/_5$ e 'l $^1/_4$, lo rimanente multiprichato per se medesmo faccia quello medesmo numero.

To the former corresponds in G [ed. Van Egmond 1978: 166]

> Truovami u'numero che tractone lo $^1/_3$ e 'l $^1/_4$, e rimanente multiprichato per se medesmo faccia ⟨12⟩.

The words and structures are so similar (for instance the use of the absolute construction "trattone", which is not compulsory) that a link within the same language area is almost certain (less certain is of course a direct derivations from Jacopo).

The third "new" example is that of C for the case $\alpha C = \beta t+n$,

> Truovamj uno numero che postovj suso 30, faccia tanto quanto multiprichato per se medesmo.

The model is obviously the same. Really new is thus only the example of G for the same case,

> Io p⟨a⟩rto 100 in una quantità e tengho a mente quello che ne viene ⟨e poi parto in 5 più che la prima volta e poi agiungo insieme quello che ne viene⟩ prima con quello che ne venne poi, e fa 20.

This type is not found in V, but is a classical problem from Arabic algebra (known already in the Old Babylonian epoch, where it is dressed as a problem about commercial rates, i.e., inverse prices).

What is interesting is the total absence of new complex dressed problems similar to those given by Jacopo, and the almost total absence of problems that in some way go beyond him. In the main, Jacopo seems to be *the* link to the tradition on which he draws. Unless the Occitan tradition was so uniform that it had practically nothing more to offer than what Jacopo had already borrowed, this is striking. As it appears, Jacopo is not only the first but also the only main contributor to *the establishment* of *abbaco* algebra.[46] On the whole, its development after his time will have been an Italian affair, and not shaped decisively by further major adoptions of Arabic (nor, probably, Catalan-Occitan) material.

There are some exceptions to this sweeping generalization. Firstly, Jacopo, as observed on p. 4, abstains from all abbreviations of mathematical terms and concepts; his is a fully rhetorical algebra, with no trace of syncopation. But not only syncopation but also the first beginnings of real

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[46] In view of the continued presence of *abbaco* writers in Montpellier this could be a hint that Jacopo did not find his material in this place but brought it to there from elsewhere. Perhaps from Catalonia? Italy is out of the question: the very impact of Jacopo's work in early fourteenth-century Italy tells that there was no community actively engaged in algebra before his time.

If Catalonia is the source, the link to the Arabic world might be the trade connections of Barcelona – cf. above. Another possibility is the *muᶜāmalat*-tradition of al-Andalus, if Ahmed Djebbar is right in suspecting that the *muᶜāmalat*-treatises of this area had absorbed algebra; this would fit the importance of "real" (*muᶜāmalat*-type) problems as illustration and the disappearance of *māl-jidr* problems.

There is a striking contrast between what happens to the algebraic and what happens to the non-algebraic problems and techniques in the decades after 1307. Some of the classical non-algebraic "recreational" problems and some commonly used techniques are absent from Jacopo's treatise (not least the "purchase of a horse" and the "hundred fowl" and the double rule of false) but soon become very common. One familiar problem seems to be there, but it is simplified in a way that changes the mathematical substance totally – namely the two birds flying from two towers of unequal height towards a water-filled bowl, where Jacopo places the bowl in the middle and asks for the difference between the distances (fol. 35^{r-v}). Soon, the orthodox and much more difficult version turns up, in which the distances are equal and the position of the bowl has to be determined.

Similarly for the double rule of false, which Jacopo either does not know or avoids deliberately in a number of problems where it would soon become the standard method: in **V.14.2**, an inverse interest problem which is instead solved by a badly explained and only approximate iteration; in **V.14.5** and **V.14.19**, both of which are solved by an intuitively limpid method that is close to the one that underlies the rather opaque double rule of false but remains different; and in **V.22.30**, apart from a very slight variation of the dress the very problem which Leonardo Fibonacci uses to introduce the trick [Boncompagni 1857: 329], and which Jacopo solves by backward reckoning. Already in the Lucca algorism from c. 1330 (the one containing **L** and **C**), there is a whole section on the "Doppia posittione", the superior efficiency of which is praised.

symbolic algebra were to arrive soon, as observed in note 24.[47] This was a development which had been well under way for quite some time in the Maghreb school of mathematics, and interaction and inspiration are not implausible. The same is likely to hold for the interest in giving names to the higher powers.

Secondly, Jacopo gives no proofs for his rules; nor do **A**, **L**, **C** and **G**, nor in fact the immense majority of fourteenth-century *abbaco* algebras. But in the long run, a number of abbacists felt the need to justify their rules, and the obvious way to do so was by means of geometry. One possibility was to exploit the vernacular translations of al-Khwārizmī and Fibonacci, which had some circulation in the fifteenth century;[48] but at times other inspirations seem to have played a role, in particular the quasi-algebraic geometric tradition of which Abū Bakr's *Liber mensurationum* and Savasorda's *Liber embadorum* are witnesses. I shall hint briefly at two indications of such borrowings:

(i) Piero della Francesca [ed. Arrighi 1970: 122], when explaining the first mixed case ($\alpha C+\beta t = n$), bases the discussion on the problem "Egl'è uno quadrato che la sua superficie, gionta coi suoi quatro lati, fa 140", and describes a diagram that does not coincide precisely with either of the ones that had gone together with this problem since its inception around 2000 BCE (the two very diagrams al-Khwārizmī uses for the same case); nor is it taken from *Elements* II.6 (nor II.4, which Cardano and Ramus would use). If not home-made (which I will not exclude), he may somehow have borrowed it from a distorted version of the quasi-algebraic tradition; he has at least borrowed extensively in the geometric part of the *Trattato d'abaco*.

(ii) Since years I have claimed that the difference between the fallacious solutions of the rectangle problem "$\sqsubset\!\sqsupset(l,w)+l+w = 62, l = w+2$" in the *Liber mensurationum* and the *Summa de arithmetica* could only have come about

[47] In the third treatise of the *Trattato dell'alcibra amuchabile* [ed. Simi 1994: 42], the sum of the two divisions in the problem just quoted from **G** is thus written

$$\frac{100}{\text{per una cosa}} \qquad \frac{100}{\text{per una cosa e più 5}}$$

which allows direct operation on the level of symbols.

[48] See [Franci & Toti Rigatelli 1985: 28*f*].

through an unknown contact to the quasi-algebraic tradition. The recent edition of Jean de Murs' *De arte mensurandi* [Busard 1998: 188] confirms this hunch. In a proposition which is so obviously isolated that it cannot be of Jean's own making he reduces it to the problem "square sides plus area given", in a way that explains both fallacies. Pacioli is not likely to have used Jean's work; but the presence of the proposition demonstrates the existence of yet another channel.

A strong suggestion of a different kind of post-Jacopo borrowing is found in Giovani di Bartolo's *Certi chasi* [ed. Pancanti 1982]. Almost 50 of its problems deal with the square root of amounts of real money, some of them also with products of such amounts (also this is a familiar Indian type); most of them are probably of Giovani's own making, but he will have constructed them according to a pre-existent pattern, for which the sole instance in Jacopo's treatise can barely have been sufficient. Though not very common, such problems remained part of the stock; one is still found in [Pacioli 1523: I, 186v].

One or the other additional problem or technique in *abbaco* algebra may turn out to be inspired from overseas after 1307. But apart from the introduction of symbolism and the naming of higher powers, none of this affected perceptibly the direction in which *abbaco* algebra developed.

If we are to characterize this direction, one aspect beyond the expansion into the higher degrees and the slow drift toward symbolization is conspicuous: the gradual disappearance of methods which *we* would call algebraic[49] but which were not regarded so by the abbacists (who instead spoke about solutions *senza regola*). As an example we may consider what happened to Jacopo's *fondaco* problems (see p. 31). I have found three other cases in which the salary of the manager of a *fondaco* is supposed to increase in geometric progression: (i) in Paolo dell'abbaco's fourteenth-century *Trattato d'aritmetica* [ed. Arrighi 1964: 149] (actually an extract from Paolo, see the incipit, ed. [Van Egmond 1980: 114]), where the increase of the salary is taken for granted (whence we may conclude that the problem was still a familiar standard problem at the moment). (ii) in Benedetto da Firenze's selection from maestro Biagio's collection of algebra problems

[49] This formulation is intended to exclude the single and double rule of false, which were only displaced later.

[ed. Pieraccini 1983: 89–91], dated before 1340. Benedetto, writing in 1463 (and commenting upon his sources), explains the presupposed increase meticulously, thereby implying that the type has disappeared since Biagio's times – indeed, after Paolo and Biagio, problems on *fattori* and *fondachi* have a different (and more realistic) mathematical structure – cf. [Tropfke/ Vogel et al 1980: 559*f*]. And (iii), in Filippo Calandri's late-fifteenth century problem collection [ed. Santini 1982: 32*f*], which actually does not deal with a *fondaco* and its manager but with the wages of a servant. It has the structure of **V.19**.3 (the most complex of all), and makes the type of increase explicit.

Paolo's simple problem (though restricted to three years) has the structure of **V.19**.2. Biagio/Benedetto's coincides with **V.19**.4 apart from a factor 2, but it is solved by means of algebra. Filippo's solution coincides in detail with Jacopo's – and so do many of his formulations. It is noteworthy that Filippo's no 44 is a three-participant analogue of **V.16**.10 (the problem involving a square root of an amount of money).

A pure-number version of Jacopo's **V.19**.4 (with sums 26 and 39, which yields a solution in integer numbers) is found in another part of Benedetto's compilation, namely in his selection from Antonio de' Mazzinghi's *Fioretti* [ed. Arrighi 1967: 5] (a disciple of Paolo dell'Abbaco). Even in this case the solution is algebraic, though different from Maestro Biagio's.

All in all, as we see, there is a tendency to replace the original solutions *senza regola* with procedures based on the standard rules of algebra when possible (which it hardly was in the case of **V.19**.3).

In the long run, there was also a tendency to return to the classical model, to reject the fallacies and the empty generalizations (the replacement of n by $\sqrt{n}$, etc.) introduced by Jacopo's successors. The conclusion of this process, as we know, led to the cancellation of the whole *abbaco* tradition from memory when algebra went into print with Pacioli and Cardano. But this is a different story, in which the Latin and "high" traditions took their revenge – as they did in sixteenth- to seventeenth-century science in general.

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