ROSKILDE UNIVERSITETSCENTER
Institut for Uddannelses-forskning, Medieforskning og Videnskabsteori

ROSKILDE UNIVERSITY CENTRE
Institute of Educational Research, Media Studies and Theory of Science

FILOSOFI OG VIDENSKABS-TEORI PÅ ROSKILDE UNIVERSITETSCENTER

3. Række: Preprints og reprints
1988 Nr. 2

ON PARTS OF PARTS AND ASCENDING CONTINUED FRACTIONS

An Investigation of the Origins and Spread of a Peculiar System

ON PARTS OF PARTS AND ASCENDING CONTINUED FRACTIONS

An Investigation of the Origins and Spread of a Peculiar System

By JENS HØYRUP

TABLE OF CONTENTS

ISLAMIC AND POST–ISLAMIC EVIDENCE

In chapter V of Leonardo Fibonacci's *Liber abaci* a number of complex writings for fractional numbers are introduced. One of them is what we call the »ascending continued fraction«, which Leonardo exemplifies by the number

$$\frac{1\ \ 5\ \ 7}{2\ \ 6\ \ 10}$$

meaning 7 10ths plus 5 6ths of a 10th plus ½ of a 6th of a 10th[1]—in more compact writing $\{^1/_{10}\cdot[7+(^1/_6)\cdot(5+½)]\}$. Others stand for $(^a/_b)\cdot(^c/_d)\cdot(^e/_f)$ (the »discontinuous fraction«), for $[(^a/_b)\cdot\{1+(^c/_d)\cdot(1+^e/_f)\}]$, and for $\{[a+(^1/_b)+(^1/_c)+(^1/_d)]/e\}$.

At least the notations for ascending continued and for discontinuous fractions were not invented by Leonardo but apparently in the Maghreb mathematical school. Both are discussed in ibn al-Bannāʾ's *Talkhīṣ aʿmāl al-ḥisāb*[2] though without indication of the way they were to be written; but various commentators show that standardized notations were in use—thus al-Qalaṣādī's *Arithmetic*[3], which explicitly requires that the denominators in an ascending continued fraction stand in descending order from the right, as it is actually the case in Leonardo's examples; and various examples in other authors, which do not all respect this canon[4].

The invention of notations was part of the general drive of Maghreb mathematics, but the ascending continued fractions and other complex fractional expressions belonged to the common heritage of Arabic mathematics. They had been amply used and discussed in the later 10th century by Abū'l-Wafāʾ in his *Book on What Scribes, Officials and the Like Need from the Science of Arithmetic*[5]. A quick search also

---

[1]Ed. Boncompagni 1857: 24. A number of later Italian occurrences until Clavius are discussed by Vogel (1982).

[2]Ed., transl. Souissi 1969: 70f.

[3]Ed. Souissi 1988: Arabic 59f, translation (with left-right inverted fraction schemes) pp. 41f.

[4]Quoted in Djebbar 1981: 46f.

[5]See Youschkevitch, "Abū'l-Wafāʾ"; *idem* 1976: 25ff; or Saidan 1974. My poor Russian has not permitted me to make much use of Medovoj's fuller description (1960) of Abū'l-Wafāʾ's textbook.

reveals their presence both in al-Khwārizmī's *Algebra*[6] from the early ninth century and in the *Liber mensurationum* by one Abū Bakr translated by Gherardo of Cremona into Latin in the 12th century and presumably written in the first place around 800 A.D.[7]. So, spot checks in Rosen's translation of al-Khwārizmī supplied the following examples:

P. 24: $^{25}/_{36}$ is transformed into »two-thirds and one-sixth of a sixth«.

P. 45: 1 *māl* is found as »a fifth and one-fifth of a fifth« of $4^1/_6$ *māl*.

P. 54: A twelfth is expressed as »the moiety of one moiety of one-third«.

P. 72: As one of several rules for finding the circular area we find the square of the diameter minus »one seventh and half one-seventh of the same«.

P. 88: The third of »nine dirhems and four-fifth of thing« is found to be »three dirhems, and one-fifth and one-third <of> one-fifth of thing«.

P. 99: »Eight-ninth of the capital less two-sevenths and two thirds of a seventh of the share of a son«.

A full but not very careful reading of the *Liber mensurationum* (which contains mostly integer numbers) revealed the following relevant passages:

N° 19 (p. 90): *7 et dimidium septime.*

N° 89 (p. 107): *43 et due quinte et quattuor quinte quinte*, resulting from the computation of $169-(11^1/_5)^2$. Similarly but in greater computational detail in N° 128 (p. 115).

N° 113 (p. 112): The root of $^3/_{16}$ *census* is expressed as *radix octave census et medietatis octave census.*

N° 144 (p. 118): The area of the circle is expressed as the square on the diameter minus *septimam et septime eius medietatem.* Similarly in N^{os} 146, 156 and 158 (pp. 119 and 124).

The elementary building stones of the ascending continued fractions are the »parts of parts«, the *partes de partibus* of the Medieval Latin tradition. The extent to which these were natural to Arabic speakers of early Islam is demonstrated in the first treatise of the *Epistles of the Brethren of Purity*, the *Rasāʾil ikhwān al-ṣafāʾ*. In this exposition of the fundaments of arithmetic great care is taken to explain case for case that the first of a collection of two is called

---

[6]Ed., transl. Rosen 1831.

[7]Ed. Busard 1968. As to the dating (built on terminological considerations), see Høyrup 1986.

a half, while the first of three is a third, that of four a fourth, and that of eleven one part of eleven; the first of twelve, however, is labeled a half of a sixths, without a single word commenting upon the reasons for or meaning of this composition. Similarly, the first of fourteen is expressed without explanation as a half of a sevenths, and that of fifteen as a third of a fifth[8].

The origin of both sorts of composite fractional expressions has been ascribed to a variety of causes, e.g. the particularities of the Arabic language. Unit fractions from $^1/_2$ to $^1/_{10}$ possess a full name of their own, while those with larger denominators require a full phrase, $^1/_n$ being »one part of *n*« or »one part of *n* parts« *unless* it can be composed from unit fractions with smaller denominators. This is indeed a good explanation that the $^1/_{14}$ of the Heronian (or rather pseudo-Heronian) rule for finding the circular area[9] becomes »half one-seventh«, and that $^1/_{25}$ becomes »one-fifth of a fifth«.

On the other hand, »the moiety of one moiety of one-third« is somewhat at odds with the hypothesis: Why not »one-third of a fourth«, when the number 12 arises as 3·4? Or at least »one-half of a sixth«, which according to Abū'l-Wafāʾ is to be preferred[10]? Furthermore, al-Khwārizmī seems to have no particular difficulty with general fractions, which abound even in those very calculations where the »parts of parts« turn up. In several cases, an expression involving »parts of parts« is simply the most easy way to state and to evaluate the immediate result of a calculation. Explanation solely from Arabic linguistic particularities seems to be ruled out, even if these have evidently tainted the way the system was used.

---

[8]Ed., transl. Brentjes 1984: 212f.

[9]»The square on the diameter minus $^1/_7$ and $^1/_{14}$ of the square« — *Geometrica* 24.40, ed., transl. Heiberg 1912: 442f. Cf. *Geometrica* 17.4, *ibid.* 332[b], 333[b].

[10]Saidan 1974: 368.

This is born out by certain older sources. One of them is the collection of arithmetical riddles in book XIV of the *Anthologia Graeca*[11]. A search through these turns out to be fascinating, since the types of fractional expressions used depend on the subject of the problem. Problems which refer to Greek mythology or history, or which deal with apples or walnuts stolen by girl friends, with the filling of jars or cisterns from several sources, with spinners', brickmakers' or gold- or silversmiths' production, or with the epochs of life—all of them make use of unit or general fractions, and none of them mention »parts of parts«.

»Parts of parts« or multiples of parts, on the other hand, turn up in the problems dealing with the Mediterranean extensions of the Silk Road (N^os^ 121 and 129), with the partition of heritages (N^os^ 128 and 143), and with the hours of the day (N^os^ 6, 139, 140, 141, and 142; N° 141 is connected to astrology). A final »fifth of a fifth« is found in N° 137, dealing with a catastrophic banquet probably meant to be held in Hellenistic Syria. It appears that a number of recreational problems belonging to different contexts (providing the dressings of the problems) have been brought together in the anthology, each conserving its own idiom for fractions. The traditional Greek idiom makes use of general and unit fractions, while the usage of the trading community and notarial calculators (and of astrologers and makers of celestial dials?) is different.

We may list the various composite fractional expressions[12]:

N° 6 (the hour of the day): »Twice two-third«.

N° 121 (travelling from Cadiz to Rome): »One-eighth and the twelfth part of one-tenth«.

N° 128 (a textually and juridically corrupt heritage): »The fifth part of seven-elevenths«.

N° 129 (travelling from Crete to Sicily): »Twice two-fifths«.

N° 137 (the Syrian banquet): »A fifth of the fifth part«.

---

[11] Ed., transl. Paton 1979.

[12] I follow Paton's translation, even though a somewhat more literal translation of certain fractional expressions *could* be made. Paton's concessions to English rhythm are immaterial for the present purpose.

N° 139 (a dial-maker asked for the hour of the day): »Four times three-fifths«.

N° 140 (the hour of a lunar eclipse): »Twice two-sixths and twice one-seventh«.
N° 141 (the hour of a birth, to be used for a horoscope): »Six times two-sevenths«.

N° 142 (The hour for spinning-women to wake up): »A fifth part of three-eighths«.

N° 143 (The heritage after a shipwrecked traveller): »Twice two-thirds«.

We observe that the usage is related to but not the same as that known from the Arabic sources. Firstly, of course, these do not contain multiples of fractions, and they would speak of »three fifths of an eighth«, not of »a fifth part of three-eighths«. Secondly, however, they mostly follow the canon made explicit by al-Qalaṣādī, taking $^1/_n$ of $^1/_m$ if $n<m$, not $^1/_m$ of $^1/_n$—and if m is 12 they split $^1/_m$ further, *viz.*, into $^1/_2$ of $^1/_6$, into $^1/_3$ of $^1/_4$ or even, as we have seen, into $^1/_2$ of $^1/_2$ of $^1/_3$.

The first deviation is probably to be explained from the recreational character of the arithmetical riddles: By being unusual, the multiples make the riddles more funny or more obscure at first sight. The demands of versification may have played a supplementary rôle—but since problems with a traditional »Greek« subject make no use of the stratagem hardly anything more than supplementary.

The second deviation, however, gives no impression of grotesquerie or supplementary obscurity. It is thus probable that it reflects the daily usage of those practitioners trading in »parts of parts«, and that *they* did not respect the Arabic canon and customs in full.

A Latin source of interest is the Carolingian collection *Propositiones ad acuendos iuvenes* conventionally ascribed to Alcuin[13]. Chronologically, it is roughly contemporary with al-Khwārizmī and probably with the *Liber mensurationum*. The material, however, appears to be inherited from late Antiquity, and the Carolingian scholar (be it Alcuin or somebody else connected to the Carolingian educational effort) has only acted as an editor.

The collection is very eclectic in character. A striking example of this is provided by the mutually contradictory techniques for (approximate) area calculation, which mix up the Old Babylonian »surveyors' formula« with the early Greek belief that isoperimetric

[13] Ed. Folkerts 1978.

figures have the same area[14]. Less paradoxical but equally diverse is the network of connections behind the arithmetical problems. N° 13, dealing with 30 successive doublings of *1*, points back to a very similar problem from Old Babylonian Mari[15] and eastward to the Indian chess-board problem and even to China. N^os^ 5, 32–34, 38–39 and 47 all belong to the type of »A hundred fowls« known from earlier Chinese and contemporary or earlier Indian sources[16] and presented by Abū Kāmil as a type of question »circulating among high-ranking and lowly people, among scholars and among the uneducated, at which they rejoice, and which they find new and beautiful; one asks the other, and he is then given an approximate and only assumed answer, they know neither principle nor rule in the matter«[17]. Other problems too point to the »oral technical literature«, the treasure of recreational problems shared and carried by the community of traders and merchants interacting along the Silk Road, the combined caravan and sea route reaching from China to Spain[18].

Connections to the *Anthologia graeca* and thus to the Greco-Roman orbit are also present. Some of them are nothing but common references to the stock of merchants' recreational problems, but N° 35 is of a different sort, *viz.*, a puzzle on heritages—one of the types, we remember, which referred to multiples of parts. According to Cantor[19], it refers to principles known from Roman jurisprudence of inheritance.

A final type represented by Nos 2, 3, 4, 40 and 45 seems to by-pass what we know from the *Anthologia graeca* and point directly to Egyptian traditions (even though matters may in reality be more complex, cf. below). Truly, when expressed in algebraic symbolism the problems in question are of a type identical with the one dominating the *Antholo-*

---

[14]See Høyrup 1987: 291 n. 38 (»an/42.9« in line 9 from bottom should read »and«).

[15]Published in Soubeyran 1984: 30. The connection and similarities between the Carolingian doublings and those from other epochs and places (except China) are discussed in detail in Høyrup 1986: 477–479. On China, see Thompson 1975: V, 542 (Z 21.1), or Høyrup 1987: 288f.

[16]See the survey in Tropfke/Vogel 1980: 613–616.

[17]My English translation from Suter 1910: 100.

[18]The classification of recreational mathematics as a parallel to folk-tales, and thus as a special *genre* of oral literature, is discussed in Høyrup 1987: 288f.

The influence of eastern trading routes on the stock from which the *Propositiones* are drawn is also made clear by problems Nos 39 and 52, dealing, respectively, with the purchase of animals (including camels) *in oriente* and with transport on camel back.

[19]1875:146–149.

gia graeca, both being represented by inhomogeneous equations of the first degree. The equations corresponding to the *Anthologia graeca*, however, are variations on the pattern

$$x\cdot(1-{}^1/_p-{}^1/_q-{}^1/_r)=R$$

(*p*, *q*, and *r* being integers), while those of the *Propositiones* build on the scheme

$$x\cdot(n+\alpha+\beta)=T$$

(*n* being an integer larger than 1 and *α* and *β* being unit fractions or »parts of parts«). The first type is similar to the Rhind Mathematical Papyrus[20], N^os^ 24–27 and 31–34, problems dealing with an unspecified quantity or »heap« (*ᶜḥᶜ*) but adding the unit fractions instead of subtracting them). The second type coincides precisely with Mathematical Papyrus Rhind N^os^ 35–38, problems dealing with the *ḥeḳat*-measure[21].

The reason for this lengthy presentation of the *Propositiones* is of course that some of its problems refer to the »parts of parts« (and two of them even come close to the scheme of ascending continued fractions):

> N° 2: *medietas medietatis, et rursus de medietate medietas* (meaning ½ of ½ and ½ of ½ of ½).
>
> N° 3: *medietas tertii.*
>
> N° 4: *medietas medietatis.*
>
> N° 40: *medietatem de medietate et de hac medietate aliam medietatem* (meaning again $^1/_4+^1/_8$).

These are four of the five problems analogous to the Egyptian *ḥeḳat*-problems. One observes that the predilection for taking successive halves instead of a simple fourth suggested by al-Khwārizmī is amply confirmed here, and is even extended to the use of $^1/_2$ of $^1/_3$ instead of $^1/_6$. This is all the more remarkable since the simple terms *quadrans* and *sextans* were at hand[22], and the composite *quarta pars* and *sexta pars* are actually used in other parts of the text (e.g. N^os^ 8 and 47). The choice of *medietas tertii* instead of *tertius medietatis* should

[20] Ed., transl. Chace et al 1929.

[21] Another group from the *Propositiones*, consisting of N^os^ 36, 44 and 48, deviates from both models but comes closest to the *ḥeḳat*-type.

[22] In the sense that the use of these subdivisions of the *as* as names for abstract fractions is described explicitly in the preface to the fifth-century *Calculus* of Victorius of Aquitania (ed. Friedlein 1871: 58f).

also be taken note of, as being in agreement with al-Qalaṣādī's canon.

BABYLONIA

Some scattered instances of »parts of parts« can thus be dug out from sources belonging to or pointing back to classical Antiquity though not to the core of Greek culture[23]. Antecedents for the systematic use of ascending continued fractions, on the other hand, must be looked for further back in time—much further, indeed.

The place in question is the Babylonian tablet MLC 1731, as it was analyzed by Abraham Sachs[24], dating from the Old Babylonian period (c. 1900 to c. 1600 B.C.; the mathematical texts belong to the second half of the period). It presents us with the following examples of composite fractions[25]:

N° 1: »One-sixth of one-fourth of [the unit] a barleycorn«.

N° 3: »One-fourth of a barleycorn and one-fourth of a fourth of a barleycorn«.

N° 4: »One-third of a barleycorn and one-eighth of a third of 20[26]«.

N° 5: »Two-thirds of 20 and one-eighth of two-thirds«.

N° 6: »A barleycorn and one-sixth of a fourth of 20«.

N° 7: »A barleycorn, two-thirds of 20 and one-eighth of two-thirds of 20.

---

[23]This peripheral status of the »parts of parts« is borne out by Ananias of Shirak's 7th century arithmetical collection (ed., transl. Kokian 1919), a work strongly dependent on contemporary Byzantine teaching. »Parts of parts« are as absent from this work as from the »Greek« problems of the *Anthologia graeca*.

[24]Sachs 1946. Besides the fractional expressions of that tablet, the article presents and discusses similar usages in other Babylonian tablets.

[25]In my translation of Babylonian texts, I follow the following conventions:

»The *n*'th part« render the expression »i g i - *n* - g á l«.

Fractions and numbers written with numerals ($^2/_3$, $^1/_2$, etc.; 1, 2, etc.) renders special cuneiform signs for these fractions and numbers.

Fractions and numbers written as words render corresponding expressions in syllabic writing.

[26]In all metrological systems, the barleycorn is 0;0,0,20 times the basic unit. »20« is thus a shorter way to write »a barleycorn«.

Nº 9: »17 bar<leycorns>, one-third of 20, and one-fourth of a third of a barleycorn«.

All these composite expressions result from the conversion of numbers belonging to the »abstract« sexagesimal system into metrological units. Sachs is certainly right in pointing out that the notation in question is used because no unit below the barleycorn existed[27]—fractions could not be expressed in terms of a smaller unit, as done in other conversions to metrological notation. Nonetheless, the tablet shows that the parlance of »parts of parts« *was* at hand, and even that there was an outspoken tendency to make use of ascending continued fractions rather that of sums of unit fractions with denominators below 10[28]. We observe that two-thirds is the only general fraction to turn up, while everything else consists of unit fractions and their combinations[29], and that al-Qalaṣādī's canon is inverted—be it accidentally or by principle.

If present nowhere else in the Old Babylonian sources, the notation *might* of course have been invented for this specific tablet. But even though it *is* rare some scattered occurrences can be found in other Old Babylonian tablets.

One instance was pointed out by Sachs: YBC 7164 Nº 7 (line 18), where the time required for a piece of work is found to be »$^2/_3$ of a day, and the 5th part of $^2/_3$ of a day«[30].

In another Yale text, »parts of parts« (though no ascending continued fractions) occur in all five times: YBC 4652 Nºs 19–22[31], problems of riddle-character dealing with the unknown weight of a stone. Here, »the 3d part of the 7th part«, »the 3d part of the 13th

---

[27]Except in the system of weights, where ½ barleycorn existed as a separate unit—cf. Sachs 1946: 208f and note 16. Most likely, however, the text is concerned with area units (among other things because the numbers converted are obtained as products of two factors, both of which vary from problem to problem).

[28]In Nº 4, the result could have been given as »$^1/_4+^1/_8$« (or as »$^1/_4+^1/_2\cdot^1/_4$«). In Nºs 5 and 7, »$^1/_2+^1/_4$« (or »$^1/_2+^1/_2\cdot^1/_2$«), and in Nº 9, »$^1/_4+^1/_6$« would have been possible.
The actual choices of the texts secure that the first member alone approximates the true value as closely as possible.

[29]Naturally enough, this reminded Sachs of the Egyptian unit fraction system (as also borrowed by the Greeks): Even there, $^2/_3$ is treated on a par with the sub-multiples $^1/_2$, $^1/_3$, $^1/_4$, etc. He did not make much of the fact that »$^1/_3$ of $^1/_5$« would be no number to an Egyptian scribe but a problem with the solution »$^1/_{15}$«. Nor was he apparently aware that much closer parallels to his notation could be found in the Arabic orbit.

[30]MCT, 82. Discussed in Sachs 1946: 212.

[31]MCT, 101.

part«, »the 3d part of the 8th part« (twice) and »$^2/_3$ of the 6th part« turn up. We observe that the ordering of factors agrees with al-Qalaṣādī's canon, and that even a »13th part« is present. (Babylonian, in contrast to Arabic, had a name for this fraction).

In the series text YBC 4714, N° 28, line 10[32] (and probably also in the damaged text of N° 27), »a half of the 3d part« turns up in the statement. This is evidently meant as a step toward greater complexity from the previous problems having »the *n*'th part« (n=7, 4, and 5) in the same place.

A text of special interest is the Susa tablet TMS V[33]. All the way through the tablet, sequences of numbers are used as abbreviations for complex numerical expressions involving parts of parts. Recurrent from section to section (albeit with some variation) is the following series (the right column gives the interpretation)

a: »2«	2
b: »3«	3
c: »4«	4 (cf. the different meaning in g)
d: »$^2/_3$«	$^2/_3$
e: »$^1/_2$«	$^1/_2$
f: »$^1/_3$«	$^1/_3$
g: »4«	$^1/_4$
h: »$^1/_3$ 4«	$^1/_3$ of $^1/_4$
i: »7«	$^1/_7$
j: »2 7«	2 times $^1/_7$
k: »7 7«	$^1/_7$ of $^1/_7$
l: »2 7 7«	2 times $^1/_7$ of $^1/_7$
m: »11«	$^1/_{11}$
n: »2 11«	2 times $^1/_{11}$
o: »11 11«	$^1/_{11}$ of $^1/_{11}$
p: »2 11 11«	2 times $^1/_{11}$ of $^1/_{11}$
q: »11 7«	$^1/_{11}$ of $^1/_7$
r: »2 11 7«	2 times $^1/_{11}$ of $^1/_7$
s: »$^2/_3$ $^1/_2$ $^1/_3$ 11 7«	$^2/_3$ of $^1/_2$ of $^1/_3$ of $^1/_{11}$ of $^1/_7$
t: »2 $^2/_3$ $^1/_2$ $^1/_3$ 11 7«	2 times $^2/_3$ of $^1/_2$ of $^1/_3$ of $^1/_{11}$ of $^1/_7$

In section 10 we also find

[32]MKT I, 490.

[33]TMS, 35–49. The tablet has probably been prepared toward the end of the Old Babylonian period.

A: »1 $^2/_3$«	1 plus $^2/_3$
B: »1 $^1/_2$«	1 plus $^1/_2$
C: »1 $^1/_3$«	1 plus $^1/_3$
D: »1 4«	1 plus $^1/_4$
E: »1 $^1/_3$ 4«	1 plus $^1/_3$ of $^1/_4$
F: »1 7«	1 plus $^1/_7$
G: »1 2 7«	1 plus 2 times $^1/_7$
H: »1 7 7«	1 plus $^1/_7$ of $^1/_7$
I: »1 2 7 7«	1 plus 2 times $^1/_7$ of $^1/_7$
J: »2 $^1/_2$«	2 plus $^1/_2$
K: »3 $^1/_3$«	3 plus $^1/_3$
L: »4 4«	4 plus $^1/_4$ (not $^1/_4$ of $^1/_4$)
M: »7 igi-7«	7 plus $^1/_7$
N: »7 2 igi-7«	7 plus 2 times $^1/_7$

In all cases, the expressions multiply the side of a square (literally: count the number of times the side is to be taken).

In order to make *his text* as unambiguous as possible, the scribe has followed a fairly strict canon, most clearly to be seen in *t* and *N*: starting from the right, we give (with increasing *denominator*) those fractions which in full writing would be written igi-*n*-gál, and which he abbreviates as *n*, an integer numeral; next come, in increasing *magnitude*, the fractions possessing their own ideogram ($^1/_3$, $^1/_2$ and $^2/_3$). This whole part of the sequence is to be understood as »parts of parts«. Then follows an eventual integer numerator (>1), and finally an eventual integer addend. As long as the numerator is never larger than 2 and the addend no larger than 1, the system is unambiguous. If we transgress these limits (as, e.g., in c and L), however, it stops being so. Inside the text, the systematic progress eliminates the ambiguities; if used as a general notation, on the other hand, the system would lead to total confusion—a fact which is obviously recognized by the scribe, since he introduces *ad hoc* the sign igi in M and N.

We must therefore presume that we are confronted with a specific, context-dependent shorthand, not with a standardized notation for general fractions, as claimed by Evert Bruins[34]. Behind the shorthand, moreover, sticks not just general fractions but the *system of »parts of parts«*, extended to include the use of numerators; the summation

[34]TMS, 36.

required to form the ascending continued fractions, on the other hand, is not visible through the notation.

In the end of the above-mentioned article discussing the Old Babylonian ascending continued fractions Sachs mentions a number of Seleucid notarial documents making use of composite expressions often involving »parts of parts« (all examples apart from N° 15 deal with the sale of temple prebends corresponding to parts of the day):

(1) »A fifth of a day and a third in (*ina*) a 15th of a day«.

(2) »A sixth, an 18th, and a 60th«.

(3) »A 30th, and a third in a 60th«.

(4) »A half in three quarters«.

(5) »A fifth in two thirds«.

(6) »Two thirds of a day and an 18th of a day«.

(7) »A sixth and a ninth of a day«.

(8) »A 20th in one day, of which a sixth in a 60th of a day is lacking«.

(9) »A 16th and a 30th of a day«, added to »a 16th of a day«, giving »an eighth and a 30th of a day«.

(10) »An eighth in a seventh«,

(11) »A half in an eighteenth«.

(12) »A third in a twelfth«.

(13) »An 18th in a seventh«.

(14) »A twelfth in a seventh«.

(15) »A half in a twelfth« (as a share of real estate).

Sachs observes in full right that the system seems less strict than the old one: No attempts is made that the first member is in itself a good approximation, and no ascending continued fractions turn up. From the present perspective, it may be of interest that all »parts of parts« *except those involving the irregular* $^{1}/_{7}$ respect al-Qalaṣādī's canon[35].

[35]»May be«, but need not. Indeed, in an article discussing some of the same examples and a number of others Denise Cocquerillat (1965) points out that the expressions are chosen in a way which will make the merchandise look as impressing as possible to a mathematically naive customer. The governing principle may thus have been sales psychology rather than any general idiomatic preference.

Its building stones being unit fractions with small denominators, the »parts of parts« scheme has often been connected to the Egyptian unit fraction system. In its mature form, as we know it from Middle Kingdom through Demotic sources, however, the Egyptian system had no predilection for small denominators. Nor were the Egyptians interested in such splittings where the first member can serve as a good first approximation—but this is precisely the point in the extension of the »parts of parts« into ascending continued fractions (as we met it already in the Old Babylonian tablet). To look for descent from the Egyptian unit fraction system is thus a red herring.

»Parts of parts« as discussed above are not common in Egypt. In fact, I only know of three places where the usage is employed *to indicate a number* (cf. below on other applications). The first of these is Rhind Mathematical Papyrus (RMP), Problem 37, one of the *ḥeḳat*-problems which were mentioned above in connection with the *Propositiones ad acuendos iuvenes*: »Go down I times 3 into the *ḥeḳat*-measure, $^1/_3$ of me is added to me, $^1/_3$ of $^1/_3$ of me is added to me, $^1/_9$ of me is added to me; return I, filled am I. Then what says it?«[36]. The second is Problem 67 of the same papyrus, »Now a herdsman came to the cattle-numbering, bringing with him 70 heads of cattle. The accountant of cattle said to the herdsman, Small indeed is the cattle-amount that thou hast brought. Where is then thy great amount of cattle? The herdsman said to him, What I have brought to thee is: $^2/_3$ of $^1/_3$ of the cattle which thou hast committed to me ...«[37]. The third, finally, belongs in the Moscow Mathematical Papyrus (MMP), Problem 20, where $2^2/_3$ is told to be *$^1/_3$ of $^2/_3$* of 20[38].

The last example is put into perspective by RMP, »Problem« 61B, which explains the method to find $^2/_3$ of any unit fraction with odd

---

[36]Chace et al 1929, Plate 59. The grammatical construction used is *$^1/_3$ n $^1/_3$*, the indirect genitive, which is also used in expressions like *$^1/_{10}$ of this 10* (RMP 28), *$^1/_2$ $^1/_4$ of a cubit* (RMP 58), *$^1/_3$ $^1/_5$ of this 30* (MMP, 3), etc. It should be distinguished from the reverse construction *z n 5*, »persons until [a total of] 5« discussed by Graefe (1979), with which it has nothing to do .

We observe that the sequence *$^1/_3$ and $^1/_3$ of $^1/_3$* suggests the idea of ascending unit fractions precisely to the same extent as did the successive *medietates* in the related *Propositiones*-problems.

[37]*Ibid.*, Plate 67. I have straightened somewhat the opaque language of the extremely literal translation.

[38]Ed., transl. Struve 1930: 95.

denominator, and uses *$^2/_3$ of $^1/_5$* as its paradigm[39]. This illustrates that a composite expression like *$^1/_5$ of $^2/_3$* would normally be ***a problem*** and no number. This it is also the way in which compositions occur in RMP, »Problem« 61, which is in fact a tabulation of a series of solutions to such problems[40].

A final place in which composite fractional expressions *α of β* turn up is the description of reversed metrological computations and conversions (RMP 44, 45, 46 and 49). As an example we may take RMP 45, which connects the two. A granary is known to contain 1500 *khar* and is supposed to have a square base of 10 cubits by 10 cubits (1 *khar* is $^2/_3$ of a cube cubit). The calculation then goes through the following steps:

1	*1500;*
$^1/_{10}$	*150;*
$^1/_{10}$ of $^1/_{10}$ of it	*15;*
$^2/_3$ of $^1/_{10}$ of $^1/_{10}$ of it:	*10*[41].

In Problem 44, which contains the direct computation, the volume is first computed as 10·10·10 [cube cubits] and then transformed into 1000+½·500=1500 *khar*. Since a solution by geometric reasoning would *start* by multiplying by $^2/_3$ (converting 1500 *khar* into 1000 cube cubit), the reversion is seen to take place at the level of computational steps and not on that of geometrical thought: The composite expressions are simply mappings of a numerical algorithm. The single constituents ($^1/_{10}$, $^1/_{10}$ and $^2/_3$) are numbers, but the compositions are neither authentic numbers nor numerical expressions to be transformed into numbers (»problems«)[42].

Though exceptional the few occurrences of composite fractional expressions *used as legitimate numbers* are sufficient proof that the fox itself and not only the diversive red herring *was* present in

[39]Chace et al 1929, Plate 83.

[40] $^2/_3$ of $^2/_3$, $^1/_3$ of $^2/_3$, $^2/_3$ of $^1/_3$, $^2/_3$ of $^1/_6$, $^2/_3$ of $^1/_2$, etc. (*loc. cit.*). Peet (1923: 103f) makes a point out of a terminological distinction inside the table, which uses the construction *α of β* in cases where α is $^2/_3$ or can be obtained from $^2/_3$ by halving or successive halvings, but a construction *β, its α* (β α . f) in other cases. Some of the formulations, indeed, have been corrected by the scribe in order to make them agree with the system; the distinction seems thus to express an actual, specific canon (which, as we observe, is broken both by the *$^1/_3$ of $^1/_3$* of RMP 37 and by the *$^1/_5$ of $^2/_3$* of MMP 20).

[41]Chace et al 1929, Plate 67.

[42]The non-numerical function of the composite expressions may also be implied by the non-observance in RMP 44, 45, 46 and 49 of the canon deduced by Peet from RMP 61: they all speak of *$^1/_{10}$ of $^1/_{10}$* (44–46 also have *$^2/_3$ of $^1/_{10}$ of $^1/_{10}$*).

Ancient Egypt: The Egyptians *were* able to understand »parts of parts« not only as problems or as sequential prescriptions but also as numbers in their own right. When would they do so?

It is difficult to deduce a rule from only three isolated instances. At least two of the present cases, however, are not isolated but participants in a specific context, on which a variety of observations can be made.

Firstly, the *ḥeḳat*-problems are formulated as riddles. When looking through the Rhind Papyrus for other riddles I only found one—*viz.*, the cattle problem in N° 67 (this is actually how I first discovered my second instance). Stylistically, these five problems are intruders into a problem collection which is otherwise written in didactically neutral style.

Secondly, the similarity was already noted between the *ḥeḳat*-problems and those problems of the *Propositiones* which make use of »parts of parts«.

Thirdly, we note that the »$^2/_3$ of $^1/_3$« of the cattle-problem is put into the mouth of the herdsman and not into that of the accountant-scribe (similarly, the »$^1/_3$ of $^1/_3$« is put into the mouth of a jug).

All this matches a comprehension of recreational mathematics as a »pure« outgrowth of practitioner's mathematics[43]. »Parts of parts« appear to have belonged to non-technical, »popular« parlance, i.e., to the very substrate from which the riddles of recreational problems were drawn. Scribal mathematics, on the other hand, made use of the highly sophisticated scheme of unit fractions; this was a technical language, and the tool which the scribe would use to solve the recreational riddles even when these were formulated in a different idiom[44].

The similarity with the Old Babylonian situation is obvious. Even here, the ascending continued fractions appeared when the result of calculations in the »technical system« of sexagesimals had to be transformed into »practical« units, while the »parts of parts« turned up in the statement of the riddles on stones of unknown weight, and when supplementary complication had to be added to purely mathematical problems.

»Parts of parts« *could* have arisen as a non-technical simplification and consecutive extension of the unit fraction system, inspired by the sequential prescriptions for reversed computational schemes. Alternatively, it could be the basis from which the unit

---

[43]See Høyrup 1987a: 2-4.

[44]The *$^1/_5$ of $^2/_3$* of MMP 20, it is true, turns up inside the calculation. It looks like a slip inspired by non-scholarly but familiar idiom.

fraction system had developed. This question cannot be decided definitively on the basis of evidence to my knowledge, but probabilities can be weighed. Much depends on the age of the full-fledged unit fraction system.

The original from which the RMP has been copied is dated to the Middle Kingdom, i.e., to the early 2nd millennium. Other papyri computing by means of the unit fraction system, some of them containing genuine accounts and not materials for teaching nor tables for reference, belong to the same period. By this time, general unit fractions had thus become a standard tool for scribal calculators[45].

Older sources, however, are almost devoid of unit fractions. Old Kingdom scribes made use of metrological sub-units and of those fractions which are *not* written in the standardized way (i.e., as $^1/_n$ written with the numeral *n* below the sign *ro*), *viz.*, $^2/_3$, $^1/_2$, and $^1/_3$[46]. Only the Fifth Dynasty Abū Sir Papyri presents us with the unit fractions $^1/_4$, $^1/_5$ and $^1/_6$[47]. At the same time, however, it presents us with striking evidence that the later *system* was not developed. The sign for $^1/_5$, indeed, appears in the connection »$^1/_5$ $^1/_5$«, meaning $^2/_5$. Later, $^2/_5$ would be no number but a problem, the solution of which was $^1/_3$+$^1/_{15}$—about one-third of the text of the Rhind Mathematical Papyrus is in fact occupied by the solution of $^2/_n$, *n* going from 3 to 101[48]. There are thus good reasons to believe that a notation for simple aliquot parts was gradually being extended toward the end of the Old Kingdom, but was not yet developed into its mature form. True, Reineke[49] thinks that it must have been needed in the complex administration of the Old Kingdom, and thus dates the development to the first three dynasties. As far as I can see, however, practical tasks are in reality better solved by means of metrological sub-units (which are standard-

---

[45]The scribal corrections in RMP 61 would suggest, however, that the canon deduced by Peet was only developed after the writing on the original, but before the copy was made some centuries later.

[46]My main basis for this description of Old Kingdom sub-unity arithmetic is the material presented in Sethe 1916.

[47]This is at least the only example I have been able to dig out with the help of competent Egyptologists. I am grateful to Professor Wolfgang Helck for referring me to the publications on the Abū Sir Papyri. The fractional signs in question are found in Posener-Kriéger & de Cenival 1968: Plates 23-25, cf. translation in Posener-Kriéger 1976 and the discussion in Silberman 1975.

[48]Silberman (1975: 249) suggests that the $^1/_5$ $^1/_5$ be explained as a product of scribal ignorance. In view of the central position occupied in Egyptian arithmetic by doubling and ensuing conversion of fractions this is almost as plausible as finding a modern accountant ignorant of the place value system.

[49]1978: 73f.

ized and can thus be marked on measuring instruments); the advantage of the unit fraction system is *theoretical*, and makes itself felt in the context of a school system.

This conclusion is supported by analysis of the pyramid problems of the RMP (N°s 56, 57, 58, 59A, 59B, 60). Those of them which appear to deal with »real«, traditional pyramids, i.e., which have a slope close to that of Old Kingdom pyramids (N°s 56–59B) measure the slope in adequate metrological units (*viz.*, palms [of horizontal retreat per cubit's ascent])[50]. The result of N° 60, which deals with some other, unidentified structure, is given as a dimensionless, abstract number. At the same time, the dimensions of the »real« pyramids are given without the unit, as it would be adequate for master-builders who knew what they were speaking about; N° 60 states the data as numbers of cubits, as suitable for a teacher instructing students who do not yet know the concrete practices and entities spoken about. It is thus likely that the author of the papyrus took over the first 5 problems with their metrological units from an older source but created or edited the final, abstract problem himself[51].

The time when teaching changed from apprenticeship to organized school teaching is fairly well-known[52]. Schools were unknown in the Old Kingdom (if we do not count the education of sons of high officials together with the royal princes), which instead relied upon an apprentice-system. Only after the break-down of the Old Kingdom do we find the first reference to a school (and the absence of a God for the school shows that schools only arose when the Pantheon had reached its definitive structure). By the time of the early Middle Kingdom, on the other hand, education is school education. There is thus a perfect coordination between the changing educational patterns, the move from metrological toward pure number, and the development of the full unit fraction system as far as it is reflected in the sources.

It is hence plausible that the systematic use of unit fractions was a fairly recent development when the original of the Rhind Papyrus was written—and implausible, as a consequence, that a non-technical usage built on »parts of parts« should already have been derived from it. On the other hand, the traces of an incipient use of the unit fraction notation in the Abū Sir Papyri fits a development starting

[50]See the comparison of real and »Rhind« slopes in Reineke 1978: 75 n. 28.

[51]This is also plausible from »a serious [conceptual] confusion [which] has taken place« in the text of N° 60, and which is pointed out and discussed by Peet (1923: 101f).

[52]See Brunner 1957: 11–15, and Wilson in Kraeling & Adams 1960: 103.

from a set of elementary aliquot parts in popular use but extending and systematizing this idiom in agreement with the requirements of school teaching.

A SCENARIO

The single occurrences of »parts of parts« and ascending continued fractions were well-established. When it comes to questions of precedence and to possible connections, many conclusions will be built on indirect evidence and on plausibility. Instead of proposing candidly a theory and claiming it to be necessary truth I will therefore suggest *a scenario*; in the final chapter I shall then try to evaluate its merits.

I shall take for granted that there existed in both Old Kingdom Egypt and in Old Babylonia a »popular« usage of elementary unit fractions (including $^2/_3$), combined into »parts of parts«; furthermore, that the Egyptians would at least make use of expressions which contained in germ the principle of ascending unit fractions, while the babylonians possessed the system in more complete form. The two systems *may* have developed independently, but since »parts of parts« appear not to be an invention near at hand in an average culture (a statement which I shall explain and qualify in the next chapter), and in particular because older Sumerian texts appear never to refer to them, I find some kind of common origin more credible. This is no impossible hypothesis. Both the Semitic (including the Babylonian) and the Ancient Egyptian languages belong to the Hamito-Semitic language family. Moreover, a socio-cultural need for simple fractions can reasonably be ascribed to the (presumably pastoral) carriers of the language before the Semitic and the Egyptian branch split from each other[53]. The extent to which common foundations have developed in mutual contact through trading routes during the fourth and third millennium B.C. is unclear: Connections existed, in all probability via Syrian territory[54], but whether they were able to influence the development of arithmetical idioms is an open question. An argument

---

[53]See the table of shared vocabulary in Diakonoff 1965: 42–49, and other common vocables mentioned elsewhere in the book. Common property is, e.g., the term *ḥsb*, "to count", "to reckon", "to calculate".

[54]See Moorey 1987 on the 4th millennium, and Klengel 1979: 61–72 on the third.

for an actual influence one way or the other (or from a common contact) could be the shared »institution« of recreational problems, which is not likely to have existed when the Semitic and Egyptian branch of the family were split no later than the fifth millennium; but since only the institution but no members (i.e., no problem-types) are shared, independent development on similar social foundations is an alternative explanation at least as near at hand. In any case, both cultures were probably too highly developed at an early stage in the domain of practical arithmetic for borrowing a usage wholesale.

The Old Babylonian »parts of parts« and ascending continued fractions are so close to the usage later testified in the Arabic sources that there is no serious reason to doubt the existence of unbroken habits in the Babylonian-Aramaic-Arabic-speaking region. (Nor is it strange that peculiarities of the single languages or the use of different computational tools or techniques would give rise to somewhat different canons and materializations of shared principles). The intense interaction of merchants along the Silk Road, which was able to carry a shared culture of recreational problems, will also have been able to spread a Semitic merchants' usage to traders and calculators of neighbouring civilizations. The early rôle of the Phoenicians and the persistent participation of Syrians and other Near Eastern merchants in Mediterranean trade, in particular, will have provided an excellent channel for the spread of the system in the West (as it was probably the channel through which a shared system of finger-reckoning spread from the Middle East to the whole Mediterranean region and as far as Bede's Northumbria[55]). The striking coincidence that problems from the *Anthologia graeca* concerned with notarial computation and with parts of the day refer to the very usage which also comes in in Seleucid calculations dealing with such subjects, as well as the references to astrology and to dial-makers in the *Anthologia*, suggests that not only traders but also juridical calculators and »Chaldean« astrologers and instrument-makers were involved in the »spread«.

»Spread«, not genuine spread. The reason that we can speak of striking coincidences is, in fact, that no real spread took place. »Parts of parts« and derived expressions are restricted to those very domains where practitioners had first employed it, using probably an idiom borrowed together with other professional instruments from the Middle East. Other domains were not affected.

---

[55]References in Høyrup 1987: 291.

The same holds for the Carolingian occurrences. Here, even the problems inspired by the eastern trade are unaffected. Only one specific type of riddle employs the usage—a type, furthermore, which ultimately points to Ancient Egypt and not to the trading network. However, Egyptian and Western Asiatic methods and traditions had been mixed up so completely during the Achaemenid and Hellenistic eras that the way from Egypt to Aachen can have passed through anywhere between Kabul and Seville. The predilection for continued halving shared with al-Khwārizmī suggests a passage through or contact with Western Asia (unless both should be connected directly and independently to Peet's conjectural canon).

EVALUATION

This was a scenario. It *could* be. *Must* it be?

Of course not. No similar reconstruction *must* be. In my exposition, however, I have explained why much evidence speaks in favour of the hypotheses. On the other hand, everything hinges on the probability of independent developments of similar structures.

»Parts of parts« may seem an idea close at hand. Everybody who understands the fractions will also understand their composition, we should think. Ascending continued fractions, furthermore, is a generalization of the metrological principle of descending sub-units; any culture possessing an ordered and multi-layered metrology should be able to invent them.

So it seems. But the actual evidence contradicts the apparent truisms. Greek Antiquity, though having demonstrably the schemes before its eyes, did not grasp at a notation which was so near at hand. It accepted it in a few select places (to where it can be assumed to have been brought). But it did not like it. For everyday use, it stuck to the Egyptian system; for mathematical purposes, it developed something like general fractions; and in astronomy, it adopted the Babylonian sexagesimal fractions.

The same holds for Latin Europe. The *Propositiones* became quite popular and influenced European recreational mathematics for centuries. But a 14th century problem coming very close to those dealing with *medietas medietatis* has »1/2 and 1/4«[56]. The usage »at hand« did not spread—on the contrary, it was resorbed.

The ascending continued fractions had a similar fate. As told above, they were taken over from Islamic arithmetic as an obligatory subject in Italian arithmetic from Leonardo onwards without acquiring ever any importance. Outside Italy, only Jordanus de Nemore tried to naturalize them as part of theoretical mathematics. He did so in his treatises on »algorism«, the computation with Hindu numerals. For this he invented the concept »dissimilar fractions«, a generalization of the notion of »consimilar fractions«—itself a generalization of the principle of sexagesimal fractions to any fixed factor of decrease. To explain what the concept was about he connected it precisely to systems of metrological sub-units[57]. Not even his closest followers, however, appear to have found anything attractive in the idea, and no echo whatsoever can be discovered.

If a concept cannot spread inside a given culture but stays restricted to a very specific use (ultimately to be resorbed) it is not likely to have been invented by this culture—especially not if there is no specific need for it in the contexts where it establishes itself. On this premise the »parts of parts« occurring in the *Anthologia graeca* and the *Propositiones* can safely be assumed to be there as the result of a borrowing. In the former case, the only conceivable source is Western Asia; in the latter, the question of the direct channel is more open, though the ultimate source is likely to be Egyptian.

Remains the question whether the Egyptian and Babylonian developments might be independent. On the premise that the creation of a scheme of »parts of parts« is empirically *not* near at hand, in spite of our *a posteriori* expectancies, it could be argued that common dependency is more likely than independent development. There is, however, a difference between the foundations on which this premise was formulated and the situation where it is now applied: The development turned out not to be near at hand *on a Greek, Latin, or Italian* linguistic background and on the background of the computational techniques and tools in common use in classical Antiquity and Medieval Europe. But eventual independent developments in Egypt and Babylonia would have taken place on structurally similar linguistic backgrounds, and maybe on the background of shared techniques and tools. The common heritage *need* not have been a developed usages of »parts of

---

[56]Ms. Columbia X 511 A13, ed. Vogel 1977: 109.

[57]See the preface to *Demonstratio de minutiis*, ed. Eneström 1913. Cf. Høyrup 1988: 337f.

parts«. It could be a system of elementary fractions and a set of linguistic usage or computational habits being naturally open to specific developments—*viz.*, the development of a scheme of »parts of parts«[58]. We cannot know whether the shared heritage was *an actual* or only *a potential* scheme (even though I find the latter possibility most plausible). And we cannot know whether an eventual shared potentiality led to independent developments or to shared acceptance of an idiom borrowed from some common contact.

---

[58]In his book (1965) on the Hamito-Semitic language family, Diakonoff mentions many instances where different languages of the family have developed similar features independently; thus as complex a phenomenon as the *pluralis fractus* (p. 68). We might speak of »structural causation«, the effect of shared linguistic structures determining that specific developments are near at hand and compatible with general linguistic customs.

»Structural causation«, however, need not be linguistic. Non-linguistic instruments for accounting and computation (be they mental or material) may in the same way open the way for specific inventions and block others which are not compatible with existing customs, tools or conceptualizations.

BIBLIOGRAPHY AND ABBREVIATIONS

Boncompagni, Baldassare (ed.), 1857. *Scritti* di Leonardo Pisano matematico del secolo decimoterzo. I. Il *Liber abaci* di Leonardo Pisano. Roma: Tipografia delle Scienze Matematiche e Fisiche.

Brentjes, Sonja, 1984. "Die erste Risâla der Rasâʾil Iḫwân as-Ṣafâʾ über elementare Zahlentheorie: Ihr mathematischer Gehalt und ihre Beziehungen zu spätantiken arithmetischen Schriften". *Janus* 71, 181–274.

Brunner, Hellmut, 1957. *Altägyptische Erziehung*. Wiesbaden: Otto Harrassowitz.

Busard, H. L. L., 1968. "L'algèbre au moyen âge: Le »Liber mensurationum« d'Abû Bekr". *Journal des Savants*, Avril–Juin 1968, 65–125.

Cantor, Moritz, 1875. *Die römischen Agrimensoren und ihre Stellung in der Geschichte der Feldmesskunst. Eine historisch-mathematische Untersuchung*. Leipzig: Teubner.

Chace, Arnold Buffum, Ludlow Bull & Henry Parker Manning, 1929. *The Rhind Mathematical Papyrus*. II. Photographs, Transcription, Transliteration, Literal Translation. *Bibliography of Egyptian and Babylonian Mathematics* (Supplement), by R. C. Archibald. *The Mathematical Leather Roll in the British Museum*, by S. R. K. Glanville. Oberlin, Ohio: Mathematical Association of America.

Cocquerillat, Denise, 1965. "Les calculs pratiques sur les fractions à l'époque seleucide". *Biblioteca Orientalis* 22, 239–242.

Diakonoff, Igor M., 1965. *Semito-Hamitic Languages*. Moskva: Nauka Publishing House.

Djebbar, A., 1981. *Enseignement et recherche mathématiques dans le Maghreb des XIIIe–XIVe siècles (étude partielle)*. (Publications mathématiques d'Orsay, 81–02). Orsay: Université de Paris-Sud.

DSB: Dictionary of Scientific Biography. 16 vols. New York: Scribner, 1970–1980.

Eneström, Georg, 1913. "Das Bruchrechnen des Nemorarius". *Bibliotheca Mathematica*, 3. Folge 14 (1913–14), 41–54.

Folkerts, Menso, 1978. "Die älteste mathematische Aufgabensammlung in lateinischer Sprache: Die Alkuin zugeschriebenen *Propositiones ad acuendos iuvenes*. Überlieferung, Inhalt, Kritische Edition". *Österreichische Akademie der Wissenschaften, Mathematisch-Naturwissenschaftliche Klasse. Denkschriften*, 116. Band, 6. Abhandlung.

Friedlein, Gottfried, 1871. "Der Calculus des Victorius". *Zeitschrift für Mathematik und Physik* 16, 42-79, 253-254, Tabelle I-IX.

Graefe, Erhart, 1979. "Ein unerkannter Gebrauch des indirekten Genitivs in Zahlkonstruktionen". Pp. 174-184 *in* M. Görg & E. Pusch (eds), *Festschrift Elmar Edel*. (Ägypten und Altes Testament. Studien zur Geschichte, Kultur und Religion Ägyptens und des Alten Testaments). Bamberg.

Heiberg, J. L. (ed., tr.), 1912. Heronis *Definitiones* cum variis collectionibus. Heronis quae feruntur *Geometrica*. (Heronis Alexandrini Opera quae supersunt omnia, IV). Leipzig: Teubner.

Høyrup, Jens, 1986. "Al-Khwârizmî, Ibn Turk, and the Liber Mensurationum: on the Origins of Islamic Algebra". *Erdem* 2:5 (Ankara), 445-484.

Høyrup, Jens, 1987. "The Formation of »Islamic Mathematics«. Sources and Conditions". *Science in Context* 1, 281-329.

Høyrup, Jens, 1987a. "Zur Frühgeschichte algebraischer Denkweisen". *Filosofi og Videnskabsteori på Roskilde Universitetscenter*. 3. Række: *Preprints og Reprints*, 1987 nr. 3. To be published in *Mathematische Semesterberichte*.

Høyrup, Jens, 1988. "Jordanus de Nemore, 13th Century Mathematical Innovator: An Essay on Intellectual Context, Achievement, and Failure". *Archive For History of Exact Sciences* 38, 307-363.

Koklan, P. Sahak (ed., tr.), 1919. "Des Anania von Schirak arithmetische Aufgaben". *Zeitschrift für die deutsch-österreichischen Gymnasien* 69 (1919-20), 112-117.

Klengel, Horst, 1979. *Handel und Händler im alten Orient*. Leipzig: Koehler und Amelang.

Kraeling, Carl, & Robert McC. Adams (eds), 1960. *City Invincible*. A Symposium on Urbanization and Cultural Development in the Ancient Near East Held at the Oriental Institute of the University of Chicago. December 4-7, 1958. Chicago: University of Chicago Press.

MCT: O. Neugebauer & A. Sachs, *Mathematical Cuneiform Texts*. (American Oriental Series, vol. 29). New Haven, Connecticut: American Oriental Society, 1945.

Medovoj, M. I., 1960. "Ob arifmetičeskom traktate Abu-l-Vafy". *Istoriko-Matematičeskie Issledovanija* 13, 253-324.

MKT: O. Neugebauer, *Mathematische Keilschrift-texte*. I-III. (Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik. Abteilung A: Quellen. 3. Band, erster-dritter Teil). Berlin: Julius Springer, 1935, 1935, 1937.

Moorey, P. R. S., 1987. "On Tracking Cultural Transfers in Prehistory: The Case of Egypt and Lower Mesopotamia in the Fourth Millennium BC". Pp. 36–46 *in* M. Rowlands, M. Larsen & K. Kristiansen (eds), *Centre and Periphery in the Ancient World.* (New Directions in Archaeology). Cambridge: Cambridge University Press.

Paton, W. R. (ed., tr.), 1979. *The Greek Anthology.* Volume V. (Loeb Classical Library). Cambridge, Mass.: Harvard University Press / London: Heinemann. 1st ed. 1918.

Peet, T. Eric, 1923. *The Rhind Mathematical Papyrus, British Museum 10057 and 10058.* Introduction, Transcription, Translation and Commentary. London: University Press of Liverpool.

Posener-Kriéger, Paule, & Jean Louis de Cenival, 1968. *Hieratic Papyri in the British Museum.* Fifth Series, *The Abu Sir Papyri.* Edited, together with Complementary Texts in Other Collections. London: The Trustees of the British Museum.

Posener-Kriéger, Paule, 1976. *Les Archives du temple funéraire de Néferirkare-kakaï (les papyrus d'Abousir).* Traduction et commentaire. 2 vols. Paris: Institut Français d'Archéologie Orientale du Caire.

Reineke, Walther Friedrich, 1978. "Gedanken zum vermutlichen Alter der mathematischen Kenntnisse im alten Ägypten". *Zeitschrift für ägyptische Sprache und Altertumskunde* 105, 67–76.

Rosen, Frederic (ed., tr.), 1831. The *Algebra* of Muhammad ben Musa, Edited and Translated. London: The Oriental Translation Fund.

Sachs, Abraham J., 1946. "Notes on Fractional Expressions in Old Babylonian Mathematical Texts". *Journal of Near Eastern Studies* 5, 203–214.

Saidan, Ahmad S., 1974. "The Arithmetic of Abū'l-Wafā'". *Isis* 65, 367–375.

Sethe, Kurt, 1916. *Von Zahlen und Zahlworten bei den Alten Ägyptern, und was für andere Völker und Sprachen daraus zu lernen ist.* (Schriften der Wissenschaftlichen gesellschaft in Straßburg, 25. Heft). Straßburg: Karl J. Trübner.

Silberman, David P., 1975. "Fractions in the Abu Sir Papyri". *Journal of Egyptian Archaeology* 61, 248–249.

Soubeyran, Denis, 1984. "Textes mathématiques de Mari". *Revue d'Assyriologie* 78, 19–48.

Souissi, Mohamed (ed., tr.), 1969. Ibn al-Bannā'. *Talkhīs aᶜmāl al-ḥisāb.* Texte établi, annoté et traduit. Tunis: L'Université de Tunis.

Souissi, Mohamed (ed., tr.), 1988. Qalaṣādī, *Kašf al-asrār ᶜan ᶜilm ḥurūf al-ġubār.* Carthage: Maison Arabe du Livre.

Struve, W. W., 1930. *Mathematischer Papyrus des Staatlichen Museums der Schönen Künste in Moskau.* (Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik. Abteilung A: Quellen. 1. Band). Berlin: Julius Springer.

Suter, Heinrich, 1910. "Das Buch der Seltenheiten der Rechenkunst von Abū Kāmil al-Miṣrī". *Bibliotheca Mathematica*, 3. Folge 11 (1910-1911), 100-120.

Thompson, Stith, 1975. *Motif-Index of Folk-Literature. A Classification of Narrative Elements in Folktales, Ballads, Myths, Fables, Mediaeval Romances, Exempla, Fabliaux, Jest Books and Local Legends.* I-VI. Rev. and enl. edition. London: Indiana University Press.

TMS: E. M. Bruins & M. Rutten, *Textes mathématiques de Suse.* (Mémoires de la Mission Archéologique en Iran, XXXIV). Paris: Paul Geuthner, 1961.

Tropfke, J./Vogel, Kurt, et al, 1980. *Geschichte der Elementarmathematik.* 4. Auflage. Band 1: *Arithmetik und Algebra.* Vollständig neu bearbeitet von Kurt Vogel, Karin Reich, Helmuth Gericke. Berlin & New York: W. de Gruyter.

Vogel, Kurt, 1977. *Ein italienisches Rechenbuch aus dem 14. Jahrhundert (Columbia X 511 A13).* (Veröffentlichungen des Deutschen Museums für die Geschichte der Wissenschaften und der Technik. Reihe C, Quellentexte und Übersetzungen, Nr. 33). München: Deutsches Museum.

Vogel, Kurt, 1982. "Zur Geschichte der Stammbrüche und der aufsteigenden Kettenbrüche". *Sudhoffs Archiv* 66, 1-19.

Youschkevitch, Adolf P., "Abū'l-Wafā'". *DSB* I, 39-43.

Youschkevitch, Adolf P., 1976. *Les mathématiques arabes (VIII^e-XV^e siècles).* (Collection d'Histoire des Sciences, 2). Paris: J. Vrin.

Author's address:

Jens Høyrup
Institute of Educational Research,
Media Studies and
Theory of Science
Roskilde University Centre
P.O. Box 260
DK-4000 Roskilde
Denmark

ISSN 0902-901X